CHAPTER 11

THE UNIFORM PLANE WAVE

In this chapter we shall apply Maxwell's equations to introduce the fundamental theory of wave motion. The uniform plane represents one of the simplest applications of Maxwell's equations, and yet it is of profound importance, since it is a basic entity by which energy is propagated. We shall explore the physical processes that determine the speed of propagation and the extent to which attenuation may occur. We shall derive and make use of the Poynting theorem to find the power carried by a wave. Finally, we shall learn how to describe wave polarization. This chapter is the foundation for our explorations in later chapters which will include wave reflection, basic transmission line and waveguiding concepts, and wave generation through antennas.

11.1 WAVE PROPAGATION IN FREE SPACE

As we indicated in our discussion of boundary conditions in the previous chapter, the solution of Maxwell's equations without the application of any boundary conditions at all represents a very special type of problem. Although we restrict our attention to a solution in rectangular coordinates, it may seem even then that we are solving several different problems as we consider various special cases in this chapter. Solutions are obtained first for free-space conditions, then for perfect dielectrics, next for lossy dielectrics, and finally for the good conductor. We do this to take advantage of the approximations that are applicable to each

special case and to emphasize the special characteristics of wave propagation in these media, but it is not necessary to use a separate treatment; it is possible (and not very difficult) to solve the general problem once and for all.

To consider wave motion in free space first, Maxwell's equations may be written in terms of **E** and **H** only as

$$\boxed{\begin{aligned}\nabla \times \mathbf{H} &= \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} &(1)\\ \nabla \times \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} &(2)\\ \nabla \cdot \mathbf{E} &= 0 &(3)\\ \nabla \cdot \mathbf{H} &= 0 &(4)\end{aligned}}$$

Now let us see whether wave motion can be inferred from these four equations without actually solving them. The first equation states that if **E** is changing with time at some point, then **H** has curl at that point and thus can be considered as forming a small closed loop linking the changing **E** field. Also, if **E** is changing with time, then **H** will in general also change with time, although not necessarily in the same way. Next, we see from the second equation that this changing **H** produces an electric field which forms small closed loops about the **H** lines. We now have once more a changing electric field, our original hypothesis, but this field is present a small distance away from the point of the original disturbance. We might guess (correctly) that the velocity with which the effect moves away from the original point is the velocity of light, but this must be checked by a more quantitative examination of Maxwell's equations.

Let us first write Maxwell's four equations above for the special case of sinusoidal (more strictly, cosinusoidal) variation with time. This is accomplished by complex notation and phasors. The procedure is identical to the one we used in studying the sinusoidal steady state in electric circuit theory.

Given the vector field

$$\mathbf{E} = E_x \mathbf{a_x}$$

we assume that the component E_x is given as

$$E_x = E(x, y, z)\cos(\omega t + \psi) \tag{5}$$

where $E(x, y, z)$ is a real function of x, y, z and perhaps ω, but not of time, and ψ is a phase angle which may also be a function of x, y, z and ω. Making use of Euler's identity,

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

we let

$$E_x + \mathrm{Re}[E(x, y, z)e^{j(\omega t+\psi)}] = \mathrm{Re}[E(x, y, z)e^{j\psi}e^{j\omega t}] \tag{6}$$

where Re signifies that the real part of the following quantity is to be taken. If we then simplify the nomenclature by dropping Re and suppressing $e^{j\omega t}$, the field quantity E_x becomes a phasor, or a complex quantity, which we identify by use of an s subscript, E_{xs}. Thus

$$E_{xs} = E(x, y, z)e^{j\psi} \tag{7}$$

and

$$\mathbf{E_s} = E_{xs}\mathbf{a_x}$$

The s can be thought of as indicating a frequency domain quantity expressed as a function of the complex frequency s, even though we shall consider only those cases in which s is a pure imaginary, $s = j\omega$.

Example 11.1

Let us express $E_y = 100\cos(10^8 t - 0.5z + 30^\circ)$ V/m as a phasor.

Solution. We first go to exponential notation,

$$E_y = \mathrm{Re}[100e^{j(10^8 t - 0.5z + 30^\circ)}]$$

and then drop Re and suppress $e^{j10^8 t}$, obtaining the phasor

$$E_{ys} = 100e^{-j0.5z + j30^\circ}$$

Note that E_y is real, but E_{ys} is in general complex. Note also that a mixed nomenclature is commonly used for the angle. That is, $0.5z$ is in radians, while 30° is in degrees.

Given a scalar component or a vector expressed as a phasor, we may easily recover the time-domain expression.

Example 11.2

Given the field intensity vector, $\mathbf{E_s} = 100\angle 30^\circ\mathbf{a_x} + 20\angle -50^\circ\mathbf{a_y} + 40\angle 210^\circ\mathbf{a_z}$ V/m, identified as a phasor by its subscript s, we desire the vector as a real function of time.

Solution. Our starting point is the phasor,

$$\mathbf{E_s} = 100\angle 30^\circ\mathbf{a_x} + 20\angle -50^\circ\mathbf{a_y} + 40\angle 210^\circ\mathbf{a_z} \text{ V/m}$$

Let us assume that the frequency is specified as 1 MHz. We first select exponential notation for mathematical clarity,

$$\mathbf{E_s} = 100e^{j30^\circ}\mathbf{a_x} + 20e^{-j50^\circ}\mathbf{a_y} + 40e^{j210^\circ}\mathbf{a_z} \text{ V/m}$$

reinsert the $e^{j\omega t}$ factor,

$$\begin{aligned}\mathbf{E_s(t)} &= (100e^{j30^\circ}\mathbf{a_x} + 20e^{-j50^\circ}\mathbf{a_y} + 40e^{j210^\circ}\mathbf{a_z})e^{j2\pi 10^6 t}\\ &= 100e^{j(2\pi 10^6 t + 30^\circ)}\mathbf{a_x} + 20e^{j(2\pi 10^6 t - 50^\circ)}\mathbf{a_y} + 40e^{j(2\pi 10^6 t + 210^\circ)}\mathbf{a_z}\end{aligned}$$

and take the real part, obtaining the real vector,

$$\mathbf{E(t)} = 100\cos(2\pi 10^6 t + 30^\circ)\mathbf{a_x} + 20\cos(2\pi 10^6 t - 50^\circ)\mathbf{a_y} + 40\cos(2\pi 10^6 t + 210^\circ)\mathbf{a_z}$$

None of the amplitudes or phase angles in this example are expressed as a function of x, y, or z, but, if any are, the same procedure is effective. Thus, if $\mathbf{H_s} = 20e^{-(0.1+j20)z}\mathbf{a_x}$ A/m, then

$$\mathbf{H(t)} = \text{Re}[20e^{-0.1z}e^{-j20z}e^{j\omega t}]\mathbf{a_x} = 20e^{-0.1z}\cos(\omega t - 20z)\mathbf{a_x} \text{ A/m}$$

Now, since

$$\frac{\partial E_x}{\partial t} = \frac{\partial}{\partial t}[E(x, y, z)\cos(\omega t + \psi)] = -\omega E(x, y, z)\sin(\omega t + \psi)$$
$$= Re[j\omega E_{xs}e^{j\omega t}]$$

it is evident that taking the partial derivative of any field quantity with respect to time is equivalent to multiplying the corresponding phasor by $j\omega$. As an example, if

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0}\frac{\partial H_y}{\partial z}$$

the corresponding phasor expression is

$$j\omega E_{xs} = -\frac{1}{\epsilon_0}\frac{\partial H_{ys}}{\partial z}$$

where E_{xs} and H_{ys} are complex quantities. We next apply this notation to Maxwell's equations. Thus, given the equation,

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

the corresponding relationship in terms of phasor-vectors is

$$\boxed{\nabla \times \mathbf{H_s} = j\omega\epsilon_0 \mathbf{E_s}} \tag{8}$$

Equation (8) and the three equations

$$\nabla \times \mathbf{E_s} = -j\omega\mu_0 \mathbf{H_s} \tag{9}$$
$$\nabla \cdot \mathbf{E_s} = 0 \tag{10}$$
$$\nabla \cdot \mathbf{H_s} = 0 \tag{11}$$

are Maxwell's four equations in phasor notation for sinusoidal time variation in free space. It should be noted that (10) and (11) are no longer independent relationships, for they can be obtained by taking the divergence of (8) and (9), respectively.

Our next step is to obtain the sinusoidal steady-state form of the wave equation, a step we could omit because the simple problem we are going to solve yields easily to simultaneous solution of the four equations above. The wave equation is an important equation, however, and it is a convenient starting point for many other investigations.

The method by which the wave equation is obtained could be accomplished in one line (using four equals signs on a wider sheet of paper):

$$\begin{aligned}\nabla \times \nabla \times \mathbf{E_s} &= \nabla(\nabla \cdot \mathbf{E_s}) - \nabla^2 \mathbf{E_s} = -j\omega\mu_0 \nabla \times \mathbf{H_s} \\ &= \omega^2 \mu_0 \epsilon_0 \mathbf{E_s} = -\nabla^2 \mathbf{E_s}\end{aligned}$$

since $\nabla \cdot \mathbf{E_s} = 0$. Thus

$$\boxed{\nabla^2 \mathbf{E_s} = -k_0^2 \mathbf{E_s}} \tag{12}$$

where k_0, the free space *wavenumber*, is defined as

$$\boxed{k_0 = \omega\sqrt{\mu_0 \epsilon_0}} \tag{13}$$

Eq. (12) is known as the vector Helmholtz equation.[1] It is fairly formidable when expanded, even in rectangular coordinates, because three scalar phasor equations result, and each has four terms. The x component of (12) becomes, still using the del-operator notation,

$$\nabla^2 \mathbf{E_{xs}} = -k_0^2 \mathbf{E_{xs}} \tag{14}$$

and the expansion of the operator leads to the second-order partial differential equation

$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} = -k_0^2 E_{xs} \tag{15}$$

Let us attempt a solution of (15) by assuming that a simple solution is possible in which E_{xs} does not vary with x or y, so that the two corresponding derivatives are zero, leading to the ordinary differential equation

$$\frac{d^2 E_{xs}}{dz^2} = -k_0^2 E_{xs} \tag{16}$$

By inspection, we may write down one solution of (16):

$$E_{xs} = E_{x0} e^{-jk_0 z} \tag{17}$$

[1] Hermann Ludwig Ferdinand von Helmholtz (1821–1894) was a professor at Berlin working in the fields of physiology, electrodynamics, and optics. Hertz was one of his students.

Next, we reinsert the $e^{j\omega t}$ factor and take the real part,

$$\boxed{E_x(z,t) = E_{x0}\cos(\omega t - k_0 z)} \tag{18}$$

where the amplitude factor, E_{x0}, is the value of E_x at $z = 0$, $t = 0$. Problem 1 at the end of the chapter indicates that

$$\boxed{E'_x(z,t) = E'_{x0}\cos(\omega t + k_0 z)} \tag{19}$$

may also be obtained from an alternate solution of the vector Helmholtz equation.

We refer to the solutions expressed in (18) and (19) as the *real instantaneous* forms of the electric field. They are the mathematical representations of what one would experimentally measure. The terms ωt and $k_0 z$, appearing in (18) and (19), have units of angle, and are usually expressed in radians. We know that ω is the radian time frequency, measuring phase shift *per unit time*, and which has units of *rad/sec*. In a similar way, we see that k_0 will be interpreted as a *spatial* frequency, which in the present case measures the phase shift *per unit distance* along the z direction. Its units are rad/m. In addition to its original name (free space wavenumber), k_0 is also the *phase constant* for a uniform plane wave in free space.

We see that the fields of (18) and (19) are x components, which we might describe as directed upward at the surface of a plane earth. The radical $\sqrt{\mu_0\epsilon_0}$, contained in k_0, has the approximate value $1/(3 \times 10^8)$ s/m, which is the reciprocal of c, the velocity of light in free space,

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.998 \times 10^8 \doteq 3 \times 10^8 \text{ m/s}$$

We can thus write $k_0 = \omega/c$, and Eq. (18), for example, can be rewritten as

$$E_x(z,t) = E_{x0}\cos[\omega(t - z/c)] \tag{20}$$

The propagation wave nature of the fields as expressed in (18), (19), and (20) can now be seen. First, suppose we were to fix the time at $t = 0$. Eq. (20) then becomes

$$E_x(z,0) = E_{x0}\cos\left(\frac{\omega z}{c}\right) = E_{x0}\cos(k_0 z) \tag{21}$$

which we identify as a simple periodic function that repeats every incremental distance λ, known as the *wavelength*. The requirement is that $k_0\lambda = 2\pi$, and so

$$\boxed{\lambda = \frac{2\pi}{k_0} = \frac{c}{f} = \frac{3 \times 10^8}{f} \qquad \text{(free space)}} \tag{22}$$

Now suppose we consider some point (such as a wave crest) on the cosine function of Eq. (21). For a crest to occur, the argument of the cosine must be an integer multiple of 2π. Considering the mth crest of the wave, the condition becomes

$$k_0 z = 2m\pi$$

So let us now consider the point on the cosine that we have chosen, and see what happens as time is allowed to increase. Eq. (18) now applies, where our requirement is that the entire cosine argument be the same multiple of 2π for all time, in order to keep track of the chosen point. From (18) and (20) our condition now becomes

$$\omega t - k_0 z = \omega(t - z/c) = 2m\pi \tag{23}$$

We see that as time increases (as it must), the position z must also increase in order to satisfy (23). *Thus the wave crest (and the entire wave) moves in the positive z direction*. The speed of travel, or wave *phase velocity*, is given by c (in free space), as can be deduced from (23). Using similar reasoning, Eq. (19), having cosine argument $(\omega t + k_0 z)$, describes a wave that moves in the *negative z* direction, since as time increases, z must now *decrease* to keep the argument constant. Waves expressed in the forms exemplified by Eqs. (18) and (19) are called *traveling waves*. For simplicity, we will restrict our attention in this chapter to only the positive z traveling wave.

Let us now return to Maxwell's equations, (8) to (11), and determine the form of the **H** field. Given $\mathbf{E_s}$, $\mathbf{H_s}$ is most easily obtained from (9),

$$\nabla \times \mathbf{E_s} = -j\omega\mu_0 \mathbf{H_s} \tag{9}$$

which is greatly simplified for a single $\mathbf{E_{xs}}$ component varying only with z,

$$\frac{dE_{xs}}{dz} = -j\omega\mu_0 H_{ys}$$

Using (17) for E_{xs}, we have

$$H_{ys} = -\frac{1}{j\omega\mu_0}(-jk_0)E_{x0}e^{-jk_0 z} = E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{-jk_0 z}$$

In real instantaneous form, this becomes

$$\boxed{H_y(z, t) = E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t - k_0 z)} \tag{24}$$

where E_{x0} is assumed real.

We therefore find the x-directed **E** field that propagates in the positive z direction is accompanied by a y-directed **H** field. Moreover, the ratio of the electric and magnetic field intensities, given by the ratio of (18) to (24),

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu_0}{\epsilon_0}} \tag{25}$$

is constant. Using the language of circuit theory, we would say that E_x and H_y are "in phase," but this in-phase relationship refers to space as well as to time. We are accustomed to taking this for granted in a circuit problem in which a

current $I_m \cos \omega t$ is assumed to have its maximum amplitude I_m throughout an entire series circuit at $t = 0$. Both (18) and (24) clearly show, however, that the maximum value of either E_x or H_y occurs when $\omega(t - z/c)$ is an integral multiple of 2π rad; neither field is a maximum everywhere at the same instant. It is remarkable, then, that the ratio of these two components, both changing in space and time, should be everywhere a constant.

The square root of the ratio of the permeability to the permittivity is called the *intrinsic impedance* η (eta),

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon}}} \tag{26}$$

where η has the dimension of ohms. The intrinsic impedance of free space is

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \doteq 120\pi\ \Omega}$$

This wave is called a *uniform plane wave* because its value is uniform throughout any plane, $z =$ constant. It represents an energy flow in the positive z direction. Both the electric and magnetic fields are perpendicular to the direction of propagation, or both lie in a plane that is transverse to the direction of propagation; the uniform plane wave is a *transverse electromagnetic wave*, or a TEM wave.

Some feeling for the way in which the fields vary in space may be obtained from Figs 11.1*a* and 11.1*b*. The electric field intensity in Fig. 11.1*a* is shown at $t = 0$, and the instantaneous value of the field is depicted along three lines, the z axis and arbitrary lines parallel to the z axis in the $x = 0$ and $y = 0$ planes. Since the field is uniform in planes perpendicular to the z axis, the variation along all three of the lines is the same. One complete cycle of the variation occurs in a wavelength, λ. The values of H_y at the same time and positions are shown in Fig. 11.1*b*.

A uniform plane wave cannot exist physically, for it extends to infinity in two dimensions at least and represents an infinite amount of energy. The distant field of a transmitting antenna, however, is essentially a uniform plane wave in some limited region; for example, a radar signal impinging on a distant target is closely a uniform plane wave.

Although we have considered only a wave varying sinusoidally in time and space, a suitable combination of solutions to the wave equation may be made to achieve a wave of any desired form. The summation of an infinite number of harmonics through the use of a Fourier series can produce a periodic wave of square or triangular shape in both space and time. Nonperiodic waves may be obtained from our basic solution by Fourier integral methods. These topics are among those considered in the more advanced books on electromagnetic theory.

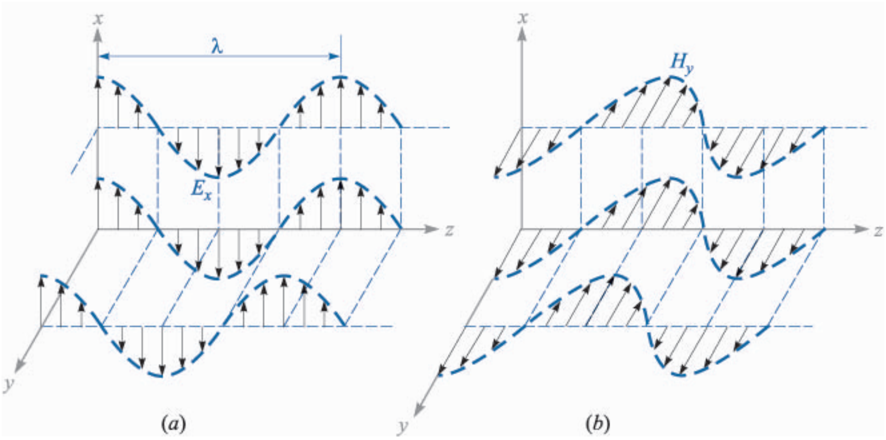

FIGURE 11.1
(*a*) Arrows represent the instantaneous values of $E_{x0}\cos[\omega(t - z/c)]$ at $t = 0$ along the z axis, along an arbitrary line in the $x = 0$ plane parallel to the z axis, and along an arbitrary line in the $y = 0$ plane parallel to the z axis. (*b*) Corresponding values of H_y are indicated. Note that E_x and H_y are in phase at any point at any time.

✔ **D11.1.** The electric field amplitude of a uniform plane wave propagating in the $\mathbf{a_z}$ direction is 250 V/m. If $\mathbf{E} = E_x \mathbf{a_x}$ and $\omega = 1.00$ Mrad/s, find: (a) the frequency; (b) the wavelength; (c) the period; (d) the amplitude of **H**.

Ans. 159 kHz; 1.88 km; 6.28 µs; 0.663 A/m

✔ **D11.2** Let $\mathbf{H_s} = (2\angle -40^\circ \mathbf{a_x} - 3\angle 20^\circ \mathbf{a_y})e^{-j0.07z}$ A/m for a uniform plane wave traveling in free space. Find: (a) ω; (b) H_x at $P(1, 2, 3)$ at $t = 31$ ns; (c) $|\mathbf{H}|$ at $t = 0$ at the origin.

Ans. 21.0 Mrad/s; 1.93 A/m; 3.22 A/m

11.2 WAVE PROPAGATION IN DIELECTRICS

Let us now extend our analytical treatment of the uniform plane wave to propagation in a dielectric of permittivity ϵ and permeability μ. The medium is isotropic and homogeneous, and the wave equation is now

$$\boxed{\nabla^2 \mathbf{E_s} = -k^2 \mathbf{E_s}} \tag{27}$$

where the wavenumber is now a function of the material properties:

$$\boxed{k = \omega\sqrt{\mu\epsilon} = k_0\sqrt{\mu_R \epsilon_R}} \tag{28}$$

For E_{xs} we have

$$\frac{d^2 E_{xs}}{dz^2} = -k^2 E_{xs} \tag{29}$$

An important feature of wave propagation in a dielectric is that k can be complex-valued, and as such is referred to as the complex *propagation constant*. A general solution of (29) in fact allows the possibility of a complex k, and it is customary to write it in terms of its real and imaginary parts in the following way:

$$\boxed{jk = \alpha + j\beta} \tag{30}$$

A solution of (29) will be:

$$\boxed{E_{xs} = E_{x0}e^{-jkz} = E_{x0}e^{-\alpha z}e^{-j\beta z}} \tag{31}$$

Multiplying (31) by $e^{j\omega t}$ and taking the real part yields a form of the field that can be more easily visualized:

$$\boxed{E_x = E_{x0}e^{-\alpha z}\cos(\omega t - \beta z)} \tag{32}$$

We recognize the above as a uniform plane wave that propagates in the forward z direction with phase constant β, but which (for positive α) loses amplitude with increasing z according to the factor $e^{-\alpha z}$. *Thus the general effect of a complex-valued k is to yield a traveling wave that changes its amplitude with distance*. If α is positive, it is called the *attenuation coefficient*. If α is negative, the wave grows in amplitude with distance, and α is called the *gain coefficient*. The latter effect would occur, for example, in laser amplifiers. In the present and future discussions in this book, we will consider only passive media, in which one or more loss mechanisms are present, thus producing a positive α.

The attenuation coefficient is measured in nepers per meter (Np/m) in order that the exponent of e be measured in the dimensionless units of nepers.[2] Thus, if $\alpha = 0.01$ Np/m, the crest amplitude of the wave at $z = 50$ m will be $e^{-0.5}/e^{-0} = 0.607$ of its value at $z = 0$. In traveling a distance $1/\alpha$ in the $+z$ direction, the amplitude of the wave is reduced by the familiar factor of e^{-1}, or 0.368.

[2] The term *neper* was selected (by some poor speller) to honor John Napier, a Scottish mathematician who first proposed the use of logarithms.

The ways in which physical processes in a material can affect the wave electric field are described through a *complex permittivity* of the form

$$\boxed{\epsilon = \epsilon' - j\epsilon''} \tag{33}$$

Two important mechanisms that give rise to a complex permittivity (and thus result in wave losses) are bound electron or ion oscillations and dipole relaxation, both of which are discussed in Appendix D. An additional mechanism is the conduction of free electrons or holes, which we will explore at length in this chapter.

Losses arising from the response of the medium to the magnetic field can occur as well, and are modeled through a *complex permeability*, $\mu = \mu' - j\mu''$. Examples of such media include *ferrimagnetic* materials, or *ferrites*. The magnetic response is usually very weak compared to the dielectric response in most materials of interest for wave propagation; in such materials $\mu \approx \mu_0$. Consequently, our discussion of loss mechanisms will be confined to those described through the complex permittivity.

We can substitute (33) into (28), which results in

$$k = \omega\sqrt{\mu(\epsilon' - j\epsilon'')} = \omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\epsilon''}{\epsilon'}} \tag{34}$$

Note the presence of the second radical factor in (34), which becomes unity (and real) as ϵ'' vanishes. With non-zero ϵ'', k is complex, and so losses occur which are quantified through the attenuation coefficient, α, in (30). The phase constant, β (and consequently the wavelength and phase velocity), will also be affected by ϵ''. α and β are found by taking the real and imaginary parts of jk from (34). We obtain:

$$\alpha = \mathrm{Re}\{jk\} = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right)^{1/2} \tag{35}$$

$$\beta = \mathrm{Im}\{jk\} = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right)^{1/2} \tag{36}$$

We see that a non-zero α (and hence loss) results if the imaginary part of the permittivity, ϵ'', is present. We also observe in (35) and (36) the presence of the ratio ϵ''/ϵ', which is called the *loss tangent*. The meaning of the term will be demonstrated when we investigate the specific case of conductive media. The practical importance of the ratio lies in its magnitude compared to unity, which enables simplifications to be made in (35) and (36).

Whether or not losses occur, we see from (32) that the wave phase velocity is given by

$$\boxed{v_p = \frac{\omega}{\beta}} \tag{37}$$

The wavelength is the distance required to effect a phase change of 2π radians

$$\beta\lambda = 2\pi$$

which leads to the fundamental definition of wavelength,

$$\boxed{\lambda = \frac{2\pi}{\beta}} \tag{38}$$

Since we have a uniform plane wave, the magnetic field is found through

$$H_{ys} = \frac{E_{x0}}{\eta} e^{-\alpha z} e^{-j\beta z}$$

where the intrinsic impedance is now a complex quantity,

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon' - j\epsilon''}} = \sqrt{\frac{\mu}{\epsilon'}} \frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}}} \tag{39}$$

The electric and magnetic fields are no longer in phase.

A special case is that of a lossless medium, or *perfect dielectric*, in which $\epsilon'' = 0$, and so $\epsilon = \epsilon'$. From (35), this leads to $\alpha = 0$, and from (36),

$$\boxed{\beta = \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu\epsilon} \qquad \text{(lossless medium)}} \tag{40}$$

With $\alpha = 0$, the real field assumes the form:

$$E_x = E_{x0} \cos(\omega t - \beta z) \tag{41}$$

We may interpret this as a wave traveling in the $+z$ direction at a phase velocity v_p, where

$$\boxed{v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu_R \epsilon_R}}}$$

The wavelength is

$$\boxed{\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\epsilon}} = \frac{1}{f\sqrt{\mu\epsilon}} = \frac{c}{f\sqrt{\mu_R\epsilon_R}} = \frac{\lambda_0}{\sqrt{\mu_R\epsilon_R}} \qquad \text{(lossless medium)}} \tag{42}$$

where λ_0 is the free space wavelength. Note that $\mu_R\epsilon_R > 1$, and therefore the wavelength is shorter and the velocity is lower in all real media than they are in free space.

Associated with E_x is the magnetic field intensity

$$H_y = \frac{E_{x0}}{\eta}\cos(\omega t - \beta z)$$

where the intrinsic impedance is

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon}}} \tag{43}$$

The two fields are once again perpendicular to each other, perpendicular to the direction of propagation, and in phase with each other everywhere. Note that when **E** is crossed into **H**, the resultant vector is in the direction of propagation. We shall see the reason for this when we discuss the Poynting vector.

Example 11.3

Let us apply these results to a 1 MHz plane wave propagating in fresh water. At this frequency, losses in water are known to be small, so for simplicity, we will neglect ϵ''. In water, $\mu_R = 1$ and at 1 MHz, $\epsilon'_R = \epsilon_R = 81$.

Solution. We begin by calculating the phase constant. Using (36) with $\epsilon'' = 0$, we have

$$\beta = \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon'_R} = \frac{\omega\sqrt{\epsilon'_R}}{c} = \frac{2\pi \times 10^6\sqrt{81}}{3.0 \times 10^8} = 0.19 \text{ rad/m}$$

Using this result, we can determine the wavelength and phase velocity:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{.19} = 33 \text{ m}$$

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^6}{.19} = 3.3 \times 10^7 \text{ m/s}$$

The wavelength in air would have been 300 m. Continuing our calculations, we find the intrinsic impedance, using (39) with $\epsilon'' = 0$:

$$\eta = \sqrt{\frac{\mu}{\epsilon'}} = \frac{\eta_0}{\sqrt{\epsilon'_R}} = \frac{377}{9} = 42\ \Omega$$

If we let the electric field intensity have a maximum amplitude of 0.1 V/m, then

$$E_x = 0.1\cos(2\pi 10^6 t - .19z) \text{ V/m}$$

$$H_y = \frac{E_x}{\eta} = 2.4 \times 10^{-3}\cos(2\pi 10^6 t - .19z) \text{ A/m}$$

✔ **D11.3.** A 9.375-GHz uniform plane wave is propagating in polyethylene (see Appendix C). If the amplitude of the electric field intensity is 500 V/m and the material is assumed to be lossless, find: (a) the phase constant; (b) the wavelength in the polyethylene; (c) the velocity of propagation; (d) the intrinsic impedance; (e) the amplitude of the magnetic field intensity.

Ans. 295 rad/m; 2.13 cm; 1.99×10^8 m/s; 251 Ω; 1.99 A/m

Example 11.4

We again consider plane wave propagation in water, but at the much higher microwave frequency of 2.5 GHz. At frequencies in this range and higher, dipole relaxation and resonance phenomena[3] in the water molecules become important. Real and imaginary parts of the permittivity are present, and both vary with frequency. At frequencies below that of visible light, the two mechanisms together produce an ϵ'' that increases with increasing frequency, reaching a local maximum in the vicinity of 10^{10} Hz. ϵ' decreases with increasing frequency. Ref. 3 provides specific details. At 2.5 GHz, dipole relaxation effects dominate. The permittivity values are $\epsilon'_R = 78$ and $\epsilon''_R = 7$. From(35), we have

$$\alpha = \frac{(2\pi \times 2.5 \times 10^9)\sqrt{78}}{(3.0 \times 10^8)\sqrt{2}}\left(\sqrt{1 + \left(\frac{7}{78}\right)^2} - 1\right)^{1/2} = 21 \text{ Np/m}$$

The first calculation demonstrates the operating principle of the *microwave oven*. Almost all foods contain water, and so can be cooked when incident microwave radiation is absorbed and converted into heat. Note that the field will attenuate to a value of e^{-1} times its initial value at a distance of $1/\alpha = 4.8$ cm. This distance is called the *penetration depth* of the material, and of course is frequency-dependent. The 4.8 cm depth is reasonable for cooking food, since it would lead to a temperature rise that is fairly uniform throughout the depth of the material. At much higher frequencies, where ϵ'' is larger, the penetration depth decreases, and too much power is absorbed at the surface; at lower frequencies, the penetration depth increases, and not enough overall absorption occurs. Commercial microwave ovens operate at frequencies in the vicinity of 2.5 GHz.

Using (36), in a calculation very similar to that for α, we find $\beta = 464$ rad/m. The wavelength is $\lambda = 2\pi/\beta = 1.4$ cm, whereas in free space this would have been $\lambda_0 = c/f = 12$ cm.

[3] These mechanisms and how they produce a complex permittivity are described in Appendix D. Additionally, the reader is referred to pp. 73–84 in Ref. 1 and pp. 678–682 in Ref. 2 for general treatments of relaxation and resonance effects on wave propagation. Discussions and data that are specific to water are presented in Ref. 3, pp. 314–316.

Using (39), the intrinsic impedance is found to be

$$\eta = \frac{377}{\sqrt{78}} \frac{1}{\sqrt{1 - j(7/78)}} = 43 + j1.9 = 43\angle 2.6^\circ \ \Omega$$

and E_x leads H_y in time by 2.6° at every point.

We next consider the case of conductive materials, in which currents are formed by the motion of free electrons or holes under the influence of an electric field. The governing relation is $\mathbf{J} = \sigma \mathbf{E}$, where σ is the material conductivity. With finite conductivity, the wave loses power through resistive heating of the material. We look for an interpretation of the complex permittivity as it relates to the conductivity. Consider the Maxwell curl equation (8) which, using (33), becomes:

$$\nabla \times \mathbf{H}_s = j\omega(\epsilon' - j\epsilon'')\mathbf{E}_s = \omega\epsilon''\mathbf{E}_s + j\omega\epsilon'\mathbf{E}_s \tag{44}$$

This equation can be expressed in a more familiar way, in which conduction current is included:

$$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega\epsilon\mathbf{E}_s \tag{45}$$

We next use $\mathbf{J}_s = \sigma\mathbf{E}_s$, and interpret ϵ in (41) as ϵ'. Eq. (45) then becomes:

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\epsilon')\mathbf{E}_s = \mathbf{J}_{\sigma s} + \mathbf{J}_{ds} \tag{46}$$

which we have expressed in terms of conduction current density, $\mathbf{J}_{\sigma s} = \sigma\mathbf{E}_s$, and displacement current density, $\mathbf{J}_{ds} = j\omega\epsilon'\mathbf{E}_s$. Comparing Eqs. (44) and (46), we find that in a conductive medium:

$$\boxed{\epsilon'' = \frac{\sigma}{\omega}} \tag{47}$$

Let us now turn our attention to the case of a dielectric material in which the loss is very small. The criterion by which we would judge whether or not the loss is small is the magnitude of the loss tangent, ϵ''/ϵ'. This parameter will have a direct influence on the attenuation coefficient, α, as seen from Eq. (35). In the case of conducting media in which (47) holds, the loss tangent becomes $\sigma/\omega\epsilon'$. By inspecting (46), we see that the ratio of condution current density to displacement current density magnitudes is

$$\frac{J_{\sigma s}}{J_{ds}} = \frac{\epsilon''}{j\epsilon'} = \frac{\sigma}{j\omega\epsilon'} \tag{48}$$

That is, these two vectors point in the same direction in space, but they are 90° out of phase in time. Displacement current density leads conduction current density by 90°, just as the current through a capacitor leads the current through a resistor in parallel with it by 90° in an ordinary electric current. This phase relationship is shown in Fig. 11.2. The angle θ (not to be confused with the polar

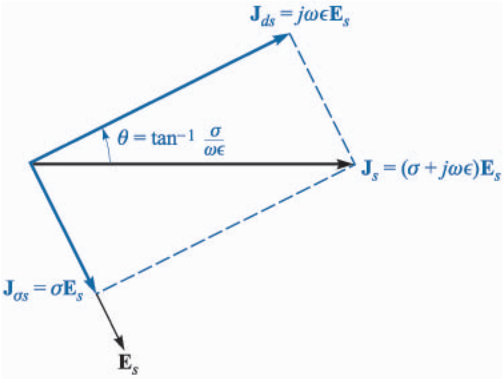

FIGURE 11.2
The time-phase relationship between $\mathbf{J}_{ds}$, $\mathbf{J}_{\sigma s}$, $\mathbf{J}_s$, and $\mathbf{E}_s$. The tangent of θ is equal to $\sigma/\omega\epsilon$, and $90^\circ - \theta$ is the common power-factor angle, or the angle by which $\mathbf{J}_s$ leads $\mathbf{E}_s$.

angle in spherical coordinates) may therefore be identified as the angle by which the displacement current density leads the total current density, and

$$\boxed{\tan\theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon'}} \tag{49}$$

The reasoning behind the term "loss tangent" is thus evident. Problem 16 at the end of the chapter indicates that the Q of a capacitor (its quality factor, not its charge) which incorporates a lossy dielectric is the reciprocal of the loss tangent.

If the loss tangent is small, then we may obtain useful approximations for the attenuation and phase constants, and the intrinsic impedance. Considering a conductive material, for which $\epsilon'' = \sigma/\omega$, (34) becomes

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\sigma}{\omega\epsilon'}} \tag{50}$$

We may expand the second radical using the binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{(2!)}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

where $|x| \ll 1$. We identify x as $-j\sigma/\omega\epsilon'$ and n as $1/2$, and thus

$$jk = j\omega\sqrt{\mu\epsilon'}\left[1 - j\frac{\sigma}{2\omega\epsilon'} + \frac{1}{8}\left(\frac{\sigma}{\omega\epsilon'}\right)^2 + \ldots\right] = \alpha + j\beta$$

Now

$$\boxed{\alpha = \mathrm{Re}(jk) \doteq j\omega\sqrt{\mu\epsilon'}\left(-j\frac{\sigma}{2\omega\epsilon'}\right) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon'}}} \tag{51}$$

and

$$\boxed{\beta = \mathrm{Im}(jk) \doteq \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\epsilon'}\right)^2\right]} \tag{52a}$$

or in many cases

$$\beta \doteq \omega\sqrt{\mu\epsilon'} \tag{52b}$$

Applying the binomial expansion to (39), we obtain

$$\eta \doteq \sqrt{\frac{\mu}{\epsilon'}}\left[1 - \frac{3}{8}\left(\frac{\sigma}{\omega\epsilon'}\right)^2 + j\frac{\sigma}{2\omega\epsilon'}\right] \tag{53a}$$

or

$$\boxed{\eta \doteq \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\sigma}{2\omega\epsilon'}\right)} \tag{53b}$$

The conditions under which the above approximations can be used depend on the desired accuracy, measured by how much the results deviate from those given by the exact formulas, (35) and (36). Deviations of no more than a few percent occur if $\sigma/\omega\epsilon' < 0.1$.

Example 11.5

As a comparison, we repeat the computations of Example 11.4, using the approximation formulas, (51), (52*b*), and (53*b*).

Solution. First, the loss tangent in this case is $\epsilon''/\epsilon' = 7/78 = 0.09$. Using (51), with $\epsilon'' = \sigma/\omega$, we have

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{1}{2}(7 \times 8.85 \times 10^{12})(2\pi \times 2.5 \times 10^9)\frac{377}{\sqrt{78}} = 21\ \mathrm{cm}^{-1}$$

We then have, using (52*b*),

$$\beta \doteq (2\pi \times 2.5 \times 10^9)\sqrt{78}/(2.99 \times 10^8) = 464\ \mathrm{rad/m}$$

Finally, with (53*b*),

$$\eta \doteq \frac{377}{\sqrt{78}}\left(1 + j\frac{7}{2 \times 78}\right) = 43 + j1.9$$

These results are identical (within the accuracy limitations as determined by the given numbers) to those of Example 11.4. Small deviations will be found, as the reader can verify by repeating the calculations of both examples and expressing the results to four or five significant figures. As we know, this latter practice would not be meaningful since the given parameters were not specified with such accuracy. Such is often the case,

since measured values are not always known with high precision. Depending on how precise these values are, one can sometimes use a more relaxed judgement on when the approximation formulas can be used, by allowing loss tangent values that can be larger than 0.1 (but still less than 1).

✔ **D11.4.** Given a nonmagnetic material having $\epsilon'_R = 3.2$ and $\sigma = 1.5 \times 10^{-4}$ S/m, find numerical values at 3 MHz for the: (a) loss tangent; (b) attenuation constant; (c) phase constant; (d) intrinsic impedance.

Ans. 0.28; 0.016 Np/m; 0.11 rad/m; $207\angle 7.8^\circ\ \Omega$

✔ **D11.5.** Consider a material for which $\mu_R = 1$, $\epsilon'_R = 2.5$, and the loss tangent is 0.12. If these three values are constant with frequency in the range 0.5 MHz $\leq f \leq$ 100 MHz, calculate: (a) σ at 1 and 75 MHz; (b) λ at 1 and 75 MHz; (c) v_p at 1 and 75 MHz.

Ans. 1.67×10^{-5} and 1.25×10^{-3} S/m; 190 and 2.53 m; 1.90×10^8 m/s twice

11.3 THE POYNTING VECTOR AND POWER CONSIDERATIONS

In order to find the power in a uniform plane wave, it is necessary to develop a theorem for the electromagnetic field known as the Poynting theorem. It was originally postulated in 1884 by an English physicist, John H. Poynting.

Let us begin with Maxwell's equation,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

and dot each side of the equation with $\mathbf{E}$,

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

We now make use of the vector identity,

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \nabla \times \mathbf{H} + \mathbf{H} \cdot \nabla \times \mathbf{E}$$

which may be proved by expansion in rectangular coordinates. Thus

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

But

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and therefore

$$-\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

or

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t}$$

However,

$$\epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\epsilon}{2} \frac{\partial E^2}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\epsilon E^2}{2} \right)$$

and

$$\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\mu H^2}{2} \right)$$

Thus

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \frac{\partial}{\partial t} \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2} \right)$$

Finally, we integrate throughout a volume,

$$-\int_{vol} \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv = \int_{vol} \mathbf{J} \cdot \mathbf{E}\, dv + \frac{\partial}{\partial t} \int_{vol} \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2} \right) dv$$

and apply the divergence theorem to obtain

$$\boxed{-\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_{vol} \mathbf{J} \cdot \mathbf{E} dv + \frac{\partial}{\partial t} \int_{vol} \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2} \right) dv} \qquad (54)$$

If we assume that there are no sources within the volume, then the first integral on the right is the total (but instantaneous) ohmic power dissipated within the volume. If sources are present within the volume, then the result of integrating over the volume of the source will be positive if power is being delivered *to* the source, but it will be negative if power is being delivered *by* the source.

The integral in the second term on the right is the total energy stored in the electric and magnetic fields,[4] and the partial derivatives with respect to time cause this term to be the time rate of increase of energy stored within this volume, or the instantaneous power going to increase the stored energy within this volume. The sum of the expressions on the right must therefore be the total power flowing *into* this volume, and thus the total power flowing *out* of the volume is

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S}$$

[4] This is the expression for magnetic field energy that we have been anticipating since Chap. 9.

where the integral is over the closed surface surrounding the volume. The cross product $\mathbf{E} \times \mathbf{H}$ is known as the Poynting vector, $\mathcal{P}$,

$$\boxed{\mathcal{P} = \mathbf{E} \times \mathbf{H}} \tag{55}$$

which is interpreted as an instantaneous power density, measured in watts per square meter ($\mathrm{W/m^2}$). This interpretation is subject to the same philosophical considerations as was the description of $(\mathbf{D} \cdot \mathbf{E})/2$ or $(\mathbf{B} \cdot \mathbf{H})/2$ as energy densities. We can show rigorously only that the integration of the Poynting vector over a closed surface yields the total power crossing the surface in an outward sense. This interpretation as a power density does not lead us astray, however, especially when applied to sinusoidally varying fields. Problem 11.18 indicates that strange results may be found when the Poynting vector is applied to time-constant fields.

The direction of the vector $\mathcal{P}$ indicates the direction of the instantaneous power flow at the point, and many of us think of the Poynting vector as a "pointing" vector. This homonym, while accidental, is correct.

Since $\mathcal{P}$ is given by the cross product of $\mathbf{E}$ and $\mathbf{H}$, the direction of power flow at any point is normal to both the $\mathbf{E}$ and $\mathbf{H}$ vectors. This certainly agrees with our experience with the uniform plane wave, for propagation in the $+z$ direction was associated with an E_x and H_y component,

$$E_x\mathbf{a_x} = H_y\mathbf{a_y} = \mathcal{P}_z\mathbf{a_z}$$

In a perfect dielectric, these $\mathbf{E}$ and $\mathbf{H}$ fields are given by

$$E_x = E_{x0}\cos(\omega t - \beta z)$$

$$H_y = \frac{E_{x0}}{\eta}\cos(\omega t - \beta z)$$

and thus

$$\mathcal{P}_z = \frac{E_{x0}^2}{\eta}\cos^2(\omega t - \beta z)$$

To find the time-average power density, we integrate over one cycle and divide by the period $T = 1/f$,

$$\begin{aligned}
\mathcal{P}_{z,av} &= \frac{1}{T}\int_0^T \frac{E_{x0}^2}{\eta}\cos^2(\omega t - \beta z)\,dt \\
&= \frac{1}{2T}\frac{E_{x0}^2}{\eta}\int_0^T [1 + \cos(2\omega t - 2\beta z)]\,dt \\
&= \frac{1}{2T}\frac{E_{x0}^2}{\eta}\left[t + \frac{1}{2\omega}\sin(2\omega t - 2\beta z)\right]_0^T
\end{aligned}$$

and

$$\mathcal{P}_{z,av} = \frac{1}{2}\frac{E_{x0}^2}{\eta} \text{ W/m}^2 \tag{56}$$

If we were using root-mean-square values instead of peak amplitudes, then the factor 1/2 would not be present.

Finally, the average power flowing through any area S normal to the z axis is[5]

$$P_{z,av} = \frac{1}{2}\frac{E_{x0}^2}{\eta} S \text{ W}$$

In the case of a lossy dielectric, E_x and H_y are not in time phase. We have

$$E_x = E_{x0}e^{-\alpha z}\cos(\omega t - \beta z)$$

If we let

$$\eta = |\eta|\angle\theta_\eta$$

then we may write the magnetic field intensity as

$$H_y = \frac{E_{x0}}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)$$

Thus,

$$\mathcal{P}_z = E_x H_y = \frac{E_{x0}^2}{|\eta|}e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)$$

Now is the time to use the identity $\cos A \cos B \equiv 1/2\cos(A+B) + 1/2\cos(A-B)$, improving the form of the last equation considerably,

$$\mathcal{P}_z = \frac{1}{2}\frac{E_{x0}^2}{|\eta|}e^{-2\alpha z}[\cos(2\omega t - 2\beta z - 2\theta_\eta) + \cos\theta_\eta]$$

We find that the power density has only a second-harmonic component and a dc component. Since the first term has a zero average value over an integral number of periods, the time-average value of the Poynting vector is

$$\mathcal{P}_{z,av} = \frac{1}{2}\frac{E_{x0}^2}{|\eta|}e^{-2\alpha z}\cos\theta_\eta$$

Note that the power density attenuates as $e^{-2\alpha z}$, whereas E_x and H_y fall off as $e^{-\alpha z}$.

We may finally observe that the above expression for $\mathcal{P}_{z,av}$ can be obtained very easily by using the phasor forms of the electric and magnetic fields:

---

[5] We shall use P for power as well as for the polarization of the medium. If they both appear in the same equation in this book, it is an error.

$$\boxed{\mathcal{P}_{z,av} = \frac{1}{2}\text{Re}(\mathbf{E_s} \times \mathbf{H_s^*}) \quad \text{W/m}^2} \tag{57}$$

where in the present case

$$\mathbf{E_s} = E_{x0}e^{-j\beta z}\mathbf{a_x}$$

and

$$\mathbf{H_s^*} = \frac{E_{x0}}{\eta^*}e^{+j\beta z}\mathbf{a_y} = \frac{E_{x0}}{|\eta|}e^{j\theta}e^{+j\beta z}\mathbf{a_y}$$

where E_{x0} has been assumed real. Eq. (57) applies to any sinusoidal electromagnetic wave, and gives both the magnitude and direction of the time-average power density.

✔ **D11.6.** At frequencies of 1, 100, and 3000 MHz, the dielectric constant of ice made from pure water has values of 4.15, 3.45, and 3.20, respectively, while the loss tangent is 0.12, 0.035, and 0.0009, also respectively. If a uniform plane wave with an amplitude of 100 V/m at $z = 0$ is propagating through such ice, find the time-average power density at $z = 0$ and $z = 10$ m for each frequency.

Ans. 27.1 and 25.7 W/m^2; 24.7 and 6.31 W/m^2; 23.7 and 8.63 W/m^2.

11.4 PROPAGATION IN GOOD CONDUCTORS: SKIN EFFECT

As an additional study of propagation with loss, we shall investigate the behavior of a good conductor when a uniform plane wave is established in it. Rather than thinking of a source embedded in a block of copper and launching a wave in that material, we should be more interested in a wave that is established by an electromagnetic field existing in some external dielectric that adjoins the conductor surface. We shall see that the primary transmission of energy must take place in the region *outside* the conductor, because all time-varying fields attenuate very quickly *within* a good conductor.

The good conductor has a high conductivity and large conduction currents. The energy represented by the wave traveling through the material therefore decreases as the wave propagates because ohmic losses are continuously present. When we discussed the loss tangent, we saw that the ratio of conduction current density to the displacement current density in a conducting material is given by $\sigma/\omega\epsilon'$. Choosing a poor metallic conductor and a very high frequency as a conservative example, this ratio[6] for nichrome ($\sigma \doteq 10^6$) at 100 MHz is about 2×10^8. Thus we have a situation where $\sigma/\omega\epsilon' \gg 1$, and we should be able to make several very good approximations to find α, β, and η for a good conductor.

[6] It is customary to take $\epsilon' = \epsilon_0$ for metallic conductors.

The general expression for the propagation constant is, from (50),

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\sigma}{\omega\epsilon'}}$$

which we immediately simplify to obtain

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{-j\frac{\sigma}{\omega\epsilon'}}$$

or

$$jk = j\sqrt{-j\omega\mu\sigma}$$

But

$$-j = 1\angle -90^\circ$$

and

$$\sqrt{1\angle -90^\circ} = 1\angle -45^\circ = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

Therefore

$$jk = j\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)\sqrt{\omega\mu\sigma}$$

or

$$jk = (j1 + 1)\sqrt{\pi f\mu\sigma} = \alpha + j\beta \tag{58}$$

Hence

$$\boxed{\alpha = \beta = \sqrt{\pi f\mu\sigma}} \tag{59}$$

Regardless of the parameters μ and σ of the conductor or of the frequency of the applied field, α and β are equal. If we again assume only an E_x component traveling in the $+z$ direction, then

$$E_x = E_{x0}e^{-z\sqrt{\pi f\mu\sigma}}\cos(\omega t - z\sqrt{\pi f\mu\sigma}) \tag{60}$$

We may tie this field in the conductor to an external field at the conductor surface. We let the region $z > 0$ be the good conductor and the region $z < 0$ be a perfect dielectric. At the boundary surface $z = 0$, (60) becomes

$$E_x = E_{x0}\cos\omega t \qquad (z = 0)$$

This we shall consider as the source field that establishes the fields within the conductor. Since displacement current is negligible,

$$\mathbf{J} = \sigma\mathbf{E}$$

Thus, the conduction current density at any point within the conductor is directly related to **E**:

$$J_x = \sigma E_x = \sigma E_{x0} e^{-z\sqrt{\pi f \mu \sigma}} \cos(\omega t - z\sqrt{\pi f \mu \sigma}) \tag{61}$$

Equations (60) and (61) contain a wealth of information. Considering first the negative exponential term, we find an exponential decrease in the conduction current density and electric field intensity with penetration into the conductor (away from the source). The exponential factor is unity at $z = 0$ and decreases to $e^{-1} = 0.368$ when

$$z = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

This distance is denoted by δ and is termed the *depth of penetration*, or the *skin depth*,

$$\boxed{\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} = \frac{1}{\beta}} \tag{62}$$

It is an important parameter in describing conductor behavior in electromagnetic fields. To get some idea of the magnitude of the skin depth, let us consider copper, $\sigma = 5.8 \times 10^7$ S/m, at several different frequencies. We have

$$\delta_{Cu} = \frac{0.066}{\sqrt{f}}$$

At a power frequency of 60 Hz, $\delta_{Cu} = 8.53$ mm, or about 1/3 in. Remembering that the power density carries an exponential term $e^{-2\alpha z}$, we see that the power density is multiplied by a factor of $0.368^2 = 0.135$ for every 8.53 mm of distance into the copper.

At a microwave frequency of 10,000 MHz, δ is 6.61×10^{-4} mm. Stated more generally, all fields in a good conductor such as copper are essentially zero at distances greater than a few skin depths from the surface. Any current density or electric field intensity established at the surface of a good conductor decays rapidly as we progress into the conductor. Electromagnetic energy is not transmitted in the interior of a conductor; it travels in the region surrounding the conductor, while the conductor merely guides the waves. We shall consider guided propagation in more detail in Chapters 13 and 14.

Suppose we have a copper bus bar in the substation of an electric utility company which we wish to have carry large currents, and we therefore select dimensions of 2 by 4 in. Then much of the copper is wasted, for the fields are greatly reduced in one skin depth, about 1/3 in.[7] A hollow conductor with a wall

[7] This utility company operates at 60 Hz.

thickness of about 1/2 in would be a much better design. Although we are applying the results of an analysis for an infinite planar conductor to one of finite dimensions, the fields are attenuated in the finite-size conductor in a similar (but not identical) fashion.

The extremely short skin depth at microwave frequencies shows that only the surface coating of the guiding conductor is important. A piece of glass with an evaporated silver surface 0.0001 in thick is an excellent conductor at these frequencies.

Next, let us determine expressions for the velocity and wavelength within a good conductor. From (62), we already have

$$\alpha = \beta = \frac{1}{\delta} = \sqrt{\pi f \mu \sigma}$$

Then, since

$$\beta = \frac{2\pi}{\lambda}$$

we find the wavelength to be

$$\boxed{\lambda = 2\pi\delta} \tag{63}$$

Also, recalling that

$$v_p = \frac{\omega}{\beta}$$

we have

$$\boxed{v_p = \omega\delta} \tag{64}$$

For copper at 60 Hz, $\lambda = 5.36$ cm and $v_p = 3.22$ m/s, or about 7.2 mi/h. A lot of us can run faster than that. In free space, of course, a 60-Hz wave has a wavelength of 3100 mi and travels at the velocity of light.

Example 11.6

Let us again consider wave propagation in water, but this time we will consider seawater. The primary difference between seawater and fresh water is of course the salt content. Sodium chloride dissociates in water to form Na^+ and Cl^- ions, which, being charged, will move when forced by an electric field. Seawater is thus conductive, and so will attenuate electromagnetic waves by this mechanism. At frequencies in the vicinity of 10^7 Hz and below, the bound charge effects in water discussed earlier are negligible, and losses in seawater arise principally from the salt-associated conductivity. We consider an incident wave of frequency 1 MHz. We wish to find the skin depth, wavelength, and phase velocity. In seawater, $\sigma = 4$ S/m, and $\epsilon'_R = 81$.

Solution. We first evaluate the loss tangent, using the given data:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{(2\pi \times 10^6)(81)(8.85 \times 10^{-12})} = 8.9 \times 10^2 \gg 1$$

Thus, seawater is a good conductor at 1 MHz (and at frequencies lower than this). The skin depth is

$$\delta = \frac{1}{\sqrt{\pi f \mu\sigma}} = \frac{1}{\sqrt{(\pi \times 10^6)(4\pi \times 10^{-7})(4)}} = 0.25 \text{ m} = 25 \text{ cm}$$

Now

$$\lambda = 2\pi\delta = 1.6 \text{ m}$$

and

$$v_p = \omega\delta = (2\pi \times 10^6)(0.25) = 1.6 \times 10^6 \text{ m/sec}$$

In free space, these values would have been $\lambda = 300$ m and of course $v = c$.

With a 25 cm skin depth, it is obvious that radio frequency communication in seawater is quite impractical. Notice however that δ varies as $1/\sqrt{f}$, so that things will improve at lower frequencies. For example, if we use a frequency of 10 Hz in the extremely low frequency (ELF) range, the skin depth is increased over that at 1 MHz by a factor of $\sqrt{10^6/10}$, so that

$$\delta(10Hz) \doteq 80\,\text{m}$$

The corresponding wavelength is $\lambda = 2\pi\delta \doteq 500$ m. Frequencies in the ELF range are in fact used for submarine communications, chiefly between gigantic ground-based antennas (required since the free-space wavelength associated with 10 Hz is 3×10^7 m) and submarines, from which a suspended wire antenna of length shorter than 500 m is sufficient to receive the signal. The drawback is that signal data rates at ELF are so slow that a single word can take several minutes to transmit. Typically, ELF signals are used to tell the submarine to implement emergency procedures, or to come near the surface in order to receive a more detailed message via satellite.

We next turn our attention to finding the magnetic field, H_y, associated with E_x. To do so, we need an expression for the intrinsic impedance of a good conductor. We begin with Eq. (39), Sec. 11.2, with $\epsilon'' = \sigma/\omega$,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon'}}$$

Since $\sigma \gg \omega\epsilon'$, we have

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}}$$

which may be written as

$$\boxed{\eta = \frac{\sqrt{2}\angle 45^\circ}{\sigma\delta} = \frac{1}{\sigma\delta} + j\frac{1}{\sigma\delta}} \tag{65}$$

Thus, if we write (60) in terms of the skin depth,

$$E_x = E_{x0} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right) \tag{66}$$

then

$$H_y = \frac{\sigma \delta E_{x0}}{\sqrt{2}} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right) \tag{67}$$

and we see that the maximum amplitude of the magnetic field intensity occurs one-eighth of a cycle later than the maximum amplitude of the electric field intensity at every point.

From (66) and (67) we may obtain the time-average Poynting vector by applying (57),

$$\mathcal{P}_{z,av} = \frac{1}{2} \frac{\sigma \delta E_{x0}^2}{\sqrt{2}} e^{-2z/\delta} \cos\left(\frac{\pi}{4}\right)$$

or

$$\boxed{\mathcal{P}_{z,av} = \frac{1}{4} \sigma \delta E_{x0}^2 e^{-2z/\delta}}$$

We again note that in a distance of one skin depth the power density is only $e^{-2} = 0.135$ of its value at the surface.

The total power loss in a width $0 < y < b$ and length $0 < x < L$ in the direction of the current, as shown in Fig. 11.3, is obtained by finding the power crossing the conductor surface within this area,

$$P_{L,av} = \int_S \mathcal{P}_{z,av} dS = \int_0^b \int_0^L \frac{1}{4} \sigma \delta E_{x0}^2 e^{-2z/\delta}|_{z=0} dx\, dy = \frac{1}{4} \sigma \delta b L E_{x0}^2$$

In terms of the current density J_{x0} at the surface,

$$J_{x0} = \sigma E_{x0}$$

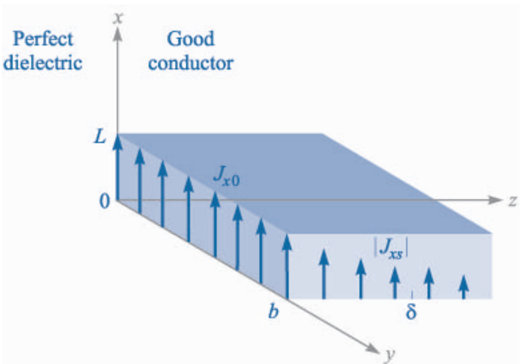

FIGURE 11.3
The current density $J_x = J_{x0} e^{-z/\delta} e^{-jz/\delta}$ decreases in magnitude as the wave propagates into the conductor. The average power loss in the region $0 < x < L$, $0 < y < b$, $z > 0$, is $\delta b L J_{x0}^2 / 4\sigma$ watts.

we have

$$P_{L,av} = \frac{1}{4\sigma}\delta b L J_{x0}^2 \tag{68}$$

Now let us see what power loss would result if the *total* current in a width b were distributed *uniformly* in one skin depth. To find the total current, we integrate the current density over the infinite depth of the conductor,

$$I = \int_0^\infty \int_0^b J_x dy\, dz$$

where

$$J_x = J_{x0} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right)$$

or in complex exponential notation to simplify the integration,

$$\begin{aligned} J_{xs} &= J_{x0} e^{-z/\delta} e^{-jz/\delta} \\ &= J_{x0} e^{-(1+j1)z/\delta} \end{aligned}$$

Therefore,

$$\begin{aligned} I_s &= \int_0^\infty \int_0^b J_{x0} e^{-(1+j1)z/\delta} dy\, dz \\ &= J_{x0} b e^{-(1+j1)z/\delta} \frac{-\delta}{1+j1}\Bigg|_0^\infty \\ &= \frac{J_{x0} b \delta}{1+j1} \end{aligned}$$

and

$$I = \frac{J_{x0} b \delta}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

If this current is distributed with a uniform density J' throughout the cross section $0 < y < b$, $0 < z < \delta$, then

$$J' = \frac{J_{x0}}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

The ohmic power loss per unit volume is $\mathbf{J} \cdot \mathbf{E}$, and thus the total instantaneous power dissipated in the volume under consideration is

$$P_L = \frac{1}{\sigma}(J')^2 b L \delta = \frac{J_{x0}^2}{2\sigma} b L \delta \cos^2\left(\omega t - \frac{\pi}{4}\right)$$

The time-average power loss is easily obtained, since the average value of the cosine-squared factor is one-half,

$$P_{L,av} = \frac{1}{4\sigma} J_{x0}^2 bL\delta \tag{69}$$

Comparing (68) and (69), we see that they are identical. Thus the average power loss in a conductor with skin effect present may be calculated by assuming that the total current is distributed uniformly in one skin depth. In terms of resistance, we may say that the resistance of a width b and length L of an infinitely thick slab with skin effect is the same as the resistance of a rectangular slab of width b, length L, and thickness δ without skin effect, or with uniform current distribution.

We may apply this to a conductor of circular cross section with little error, provided that the radius a is much greater than the skin depth. The resistance at a high frequency where there is a well-developed skin effect is therefore found by considering a slab of width equal to the circumference $2\pi a$ and thickness δ. Hence

$$R = \frac{L}{\sigma S} = \frac{L}{2\pi a \sigma \delta} \tag{70}$$

A round copper wire of 1 mm radius and 1 km length has a resistance at direct current of

$$R_{dc} = \frac{10^3}{\pi 10^{-6}(5.8 \times 10^7)} = 5.48\ \Omega$$

At 1 MHz, the skin depth is 0.066 mm. Thus $\delta \ll a$, and the resistance at 1 MHz is found by (70),

$$R = \frac{10^3}{2\pi 10^{-3}(5.8 \times 10^7)(0.066 \times 10^{-3})} = 41.5\ \Omega$$

✔ **D11.7.** A steel pipe is constructed of a material for which $\mu_R = 180$ and $\sigma = 4 \times 10^6$ S/m. The two radii are 5 and 7 mm, and the length is 75 m. If the total current $I(t)$ carried by the pipe is $8 \cos \omega t$ A, where $\omega = 1200\pi$ rad/s, find: (a) the skin depth; (b) the effective resistance; (c) the dc resistance; (d) the time-average power loss.

Ans. 0.766 mm; 0.557 Ω; 0.249 Ω; 17.82 W

11.5 WAVE POLARIZATION

In the previous sections, we have treated uniform plane waves in which the electric and magnetic field vectors were assumed to lie in fixed directions. Specifically, with the wave propagating along the z axis, **E** was taken to lie along x, which then required **H** to lie along y. This orthogonal relationship between **E**, **H**, and $\mathcal{P}$ is always true for a uniform plane wave. The directions of **E** and **H** within the plane perpendicular to $\mathbf{a}_z$ may change, however, as functions of time and position, depending on how the wave was generated, or on what type of medium it is propagating through. Thus a complete description of

an electromagnetic wave would not only include parameters such as its wavelength, phase velocity, and power, but also a statement of the instantaneous orientation of its field vectors. *The wave polarization is defined as its electric field vector orientation as a function of time, at a fixed position in space.* Specifying only the electric field direction is sufficient, since magnetic field is readily found from **E** using Maxwell's equations.

In the waves we have previously studied, **E** was in a fixed straight orientation for all times and positions. Such a wave is said to be *linearly polarized*. We have taken **E** to lie along the x axis, but the field could be oriented in any fixed direction in the $x-y$ plane and be linearly polarized. For positive z-propagation, the wave would generally have its electric field phasor expressed as

$$\mathbf{E_s} = (E_{x0}\mathbf{a_x} + E_{y0}\mathbf{a_y})e^{-\alpha z}e^{-j\beta z} \tag{71}$$

where E_{x0} and E_{y0} are constant amplitudes along x and y. The associated magnetic field is readily found by determining its x and y components directly from those of $\mathbf{E_s}$. Specifically, $\mathbf{H_s}$ for the wave of Eq. (71) is

$$\mathbf{H_s} = [H_{x0}\mathbf{a_x} + H_{y0}\mathbf{a_y}]e^{-\alpha z}e^{-j\beta z} = \left[-\frac{E_{y0}}{\eta}\mathbf{a_x} + \frac{E_{x0}}{\eta}\mathbf{a_y}\right]e^{-\alpha z}e^{-j\beta z} \tag{72}$$

The two fields are sketched in Fig. 11.4. The figure demonstrates the reason for the minus sign in the term involving E_{y0} in Eq. (72). The direction of power flow, given by $\mathbf{E}\times\mathbf{H}$, is in the positive z direction in this case. A component of **E** in the positive y direction would require a component of **H** in the negative x direction—thus the minus sign. Using (71) and (72), the power density in the wave is found using (57):

$$\begin{aligned}\mathcal{P}_{z,av} &= \frac{1}{2}\mathrm{Re}\{\mathbf{E_s}\times\mathbf{H_s^*}\} = \frac{1}{2}\mathrm{Re}\{E_{x0}H_{y0}^*(\mathbf{a_x}\times\mathbf{a_y}) + E_{y0}H_{x0}^*(\mathbf{a_y}\times\mathbf{a_x})\}e^{-2\alpha z}\\ &= \frac{1}{2}\mathrm{Re}\left\{\frac{E_{x0}E_{x0}^*}{\eta^*} + \frac{E_{y0}E_{y0}^*}{\eta^*}\right\}e^{-2\alpha z}\mathbf{a_z} = \frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta^*}\right\}(|E_{x0}|^2 + |E_{y0}|^2)e^{-2\alpha z}\mathbf{a_z}\ \mathrm{W/m^2}\end{aligned}$$

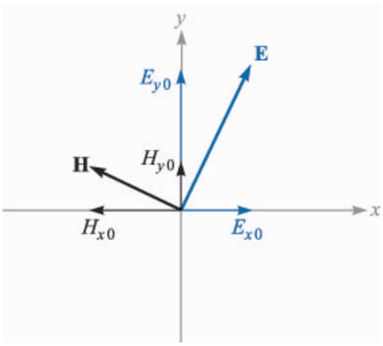

FIGURE 11.4
Electric and magnetic field configuration for a general linearly polarized plane wave, propagating in the forward z direction (out of the page). Field components correspond to those in Eqs. (71) and (72).

This result demonstrates the idea that our linearly polarized plane wave can be considered as two distinct plane waves having x and y polarizations, and whose electric fields are adding *in phase* to produce the total **E**. The same is true for the magnetic field components. This is a critical point in understanding wave polarization, in that *any polarization state can be described in terms of mutually perpendicular components of the electric field and their relative phasing*.

We next consider the effect of a phase difference, ϕ, between E_{x0} and E_{y0}, where $\phi < \pi/2$. For simplicity, we will consider propagation in a lossless medium. The total field in phasor form is

$$\mathbf{E_s} = (E_{x0}\mathbf{a_x} + E_{y0}e^{j\phi}\mathbf{a_y})e^{-j\beta z} \tag{73}$$

Again, to aid in visualization, we convert this wave to real instantaneous form by multiplying by $e^{j\omega t}$ and taking the real part:

$$\mathbf{E}(z, t) = E_{x0}\cos(\omega t - \beta z)\mathbf{a_x} + E_{y0}\cos(\omega t - \beta z + \phi)\mathbf{a_y} \tag{74}$$

where we have assumed that E_{x0} and E_{y0} are real. Suppose we set $t = 0$, in which case (74) becomes (using $\cos(-x) = \cos(x)$)

$$\mathbf{E}(z, 0) = E_{x0}\cos(\beta z)\mathbf{a_x} + E_{y0}\cos(\beta z - \phi)\mathbf{a_y} \tag{75}$$

The component magnitudes of $\mathbf{E}(z, 0)$ are plotted as functions of z in Fig. 11.5. Since time is fixed at zero, the wave is frozen in position. An observer can move along the z axis, measuring the component magnitudes and thus the orientation of the total electric field at each point. Let's consider a crest of E_x, indicated as point a in the figure. If ϕ were zero, E_y would have a crest at the same location. Since ϕ is not zero (and positive), the crest of E_y that would otherwise occur at point a is now displaced to point b further down z. The two points are separated by distance ϕ/β. E_y thus *lags behind* E_x when considering the *spatial* dimension.

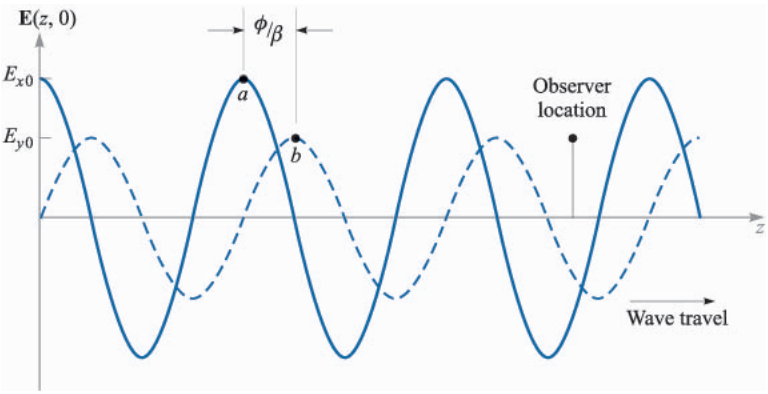

FIGURE 11.5
Plots of the electric field component magnitudes in Eq. (75) as functions of z. Note that the y component lags the x component in z. As time increases, both waves travel to the right, as per Eq. (74). Thus to an observer at a fixed location, the y component leads in time.

Now suppose the observer stops at some location on the z axis, and time is allowed to move forward. Both fields now move in the positive z direction, as (74) indicates. But point b reaches the observer first, followed by point a. So we see that E_y *leads* E_x when considering the *time* dimension. In either case (fixed t and varying z, or vice-versa) the observer notes that the net field rotates about the z axis while its magnitude changes. Considering a starting point in z and t, at which the field has a given orientation and magnitude, the wave will return to the same orientation and magnitude at a distance of one wavelength in z (for fixed t) or at a time $t = 2\pi/\omega$ later (at a fixed z).

For illustration purposes, if we take the length of the field vector as a measure of its magnitude, we would find that at a fixed position, the tip of the vector would trace out the shape of an ellipse over time $t = 2\pi/\omega$. The wave is thus said to be *elliptically polarized*. Elliptical polarization is in fact the most general polarization state of a wave, since it encompasses any magnitude and phase difference between E_x and E_y. Linear polarization is a special case of elliptical polarization, in which the phase difference is zero.

Another special case of elliptical polarization occurs when $E_{x0} = E_{y0} = E_0$ and when $\phi = \pm\pi/2$. The wave in this case exhibits *circular polarization*. To see this, we incorporate the above restrictions into Eq. (74) to obtain

$$\begin{aligned}\mathbf{E}(z, t) &= E_0[\cos(\omega t - \beta z)\mathbf{a_x} + \cos(\omega t - \beta z \pm \pi/2)\mathbf{a_y}] \\ &= E_0[\cos(\omega t - \beta z)\mathbf{a_x} \mp \sin(\omega t - \beta z)\mathbf{a_y}]\end{aligned} \tag{76}$$

If we consider a fixed position along z (such as $z = 0$) and allow time to vary, (76), with $\phi = +\pi/2$, becomes

$$\mathbf{E}(0, t) = E_0[\cos(\omega t)\mathbf{a_x} - \sin(\omega t)\mathbf{a_y}] \tag{77}$$

If we choose $\phi = -\pi/2$ in (76), we obtain

$$\mathbf{E}(0, t) = E_0[\cos(\omega t)\mathbf{a_x} + \sin(\omega t)\mathbf{a_y}] \tag{78}$$

The field vector of Eq. (78) rotates in the counter-clockwise direction in the x, y plane, while maintaining constant amplitude E_0, and so the tip of the vector traces out a circle. Fig. 11.6 shows this behavior. Choosing $\phi = +\pi/2$ leads to (77), whose field vector rotates in the clockwise direction. The *handedness* of the circular polarization is associated with the rotation and propagation directions in the following manner: The wave exhibits *left circular polarization* (l.c.p.) if when orienting the left hand with the thumb in the direction of propagation, the fingers curl in the rotation direction of the field with time. The wave exhibits *right circular polarization* (r.c.p.) if with the right hand thumb in the propagation direction, the fingers curl in the field rotation direction. Thus, with forward z propagation, (77) describes a left circularly polarized wave, and (78) a right circularly polarized wave. The same convention is applied to elliptical polarization, in which the descriptions *left elliptical polarization* and *right elliptical polarization* are used.

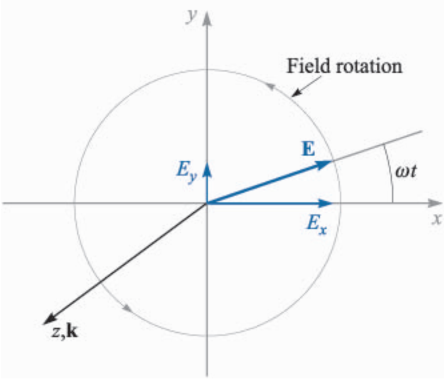

FIGURE 11.6
Electric field in the x, y plane of a right circularly polarized plane wave, as described by Eq. (78). As the wave propagates in the forward z direction, the field vector rotates counterclockwise in the x, y plane.

Using (76), the instantaneous angle of the field from the x direction can be found for any position along z through

$$\theta(z, t) = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}\left(\frac{\mp\sin(\omega t - \beta z)}{\cos(\omega t - \beta z)}\right) = \mp(\omega t - \beta z) \tag{79}$$

where again the minus sign (yielding l.c.p. for positive z travel) applies for the choice of $\phi = +\pi/2$ in (76); the plus sign (yielding r.c.p. for positive z travel) is used if $\phi = -\pi/2$. If we choose $z = 0$, the angle becomes simply ωt, which reaches 2π (one complete rotation) at time $t = 2\pi/\omega$. If we chose $t = 0$ and allow z to vary, we form a corkscrew-like field pattern. One way to visualize this is to consider a spiral staircase-shaped pattern, in which the field lines (stair-steps) are perpendicular to the z (or staircase) axis. The relationship between this spatial field pattern and the resulting time behavior at fixed z as the wave propagates is shown in an artist's conception on the front cover.[8] Changing handedness of the polarization is accomplished by reversing the pitch of the corkscrew pattern. The spiral staircase model is only a visualization aid. It must be remembered that the wave is still a uniform plane wave, whose fields at any position along z are infinite in extent over the transverse plane.

There are many uses of circularly polarized waves. Perhaps the most obvious advantage is that reception of a wave having circular polarization does not depend on the antenna orientation in the plane normal to the propagation direction. Dipole antennas, for example, are required to be oriented along the electric field direction of the signal they receive. If circularly polarized signals are transmitted, the receiver orientation requirements are relaxed considerably. In optics, circularly polarized light can be passed through a polarizer of any orientation, thus yielding linearly polarized light in any direction (although one loses half the original power this way). Other uses involve treating linearly

[8] The artistic license enabling the cover wave to cast a shadow of itself may cause one to stop and think for a moment (but only for a moment).

polarized light as a superposition of circularly polarized waves, to be described below.

Circularly polarized light can be generated using an *anisotropic* medium—a material whose permittivity is a function of electric field direction. Many crystals have this property. A crystal orientation can be found such that along one direction (say, the x axis), the permittivity is lowest, while along the orthogonal direction (y axis), the permittivity is highest. The strategy is to input a linearly polarized wave with its field vector at 45 degrees to the x and y axes of the crystal. It will thus have equal-amplitude x and y components in the crystal, and these will now propagate in the z direction at different speeds. A phase difference (or *retardation*) accumulates between the components as they propagate, which can reach $\pi/2$ if the crystal is long enough. The wave at the output thus becomes circularly polarized. Such a crystal, cut to the right length, and used in this manner, is called a *quarter-wave plate*, since it introduces a relative phase shift of $\pi/2$ between E_x and E_y, which is equivalent to $\lambda/4$.

It is useful to express circularly polarized waves in phasor form. To do this, we note that (76) can be expressed as

$$\mathbf{E}(z,t) = \mathrm{Re}\left\{E_0 e^{j\omega t} e^{-j\beta z}\left[\mathbf{a_x} + e^{\pm j\pi/2}\mathbf{a_y}\right]\right\}$$

Using the fact that $e^{\pm j\pi/2} = \pm j$, we identify the phasor form as:

$$\boxed{\mathbf{E_s} = E_0(\mathbf{a_x} \pm j\mathbf{a_y})e^{-j\beta z}} \tag{80}$$

where the plus sign is used for left circular polarization, and the minus sign for right circular polarization. If the wave propagates in the negative z direction, we would have

$$\boxed{\mathbf{E_s} = E_0(\mathbf{a_x} \pm j\mathbf{a_y})e^{+j\beta z}} \tag{81}$$

where in this case the positive sign applies to right circular polarization, and the minus sign to left circular polarization. The student is encouraged to verify this.

Example 11.7

Let us consider the result of superimposing left and right circularly polarized fields of the same amplitude, frequency, and propagation direction, but where a phase shift of δ radians exists between the two.

Solution. Taking the waves to propagate in the $+z$ direction, and introducing a relative phase, δ, the total phasor field is found, using (80):

$$\mathbf{E}_s^T = \mathbf{E}_s^R + \mathbf{E}_s^L = E_0[\mathbf{a_x} - j\mathbf{a_y}]e^{-j\beta z} + E_0[\mathbf{a_x} + j\mathbf{a_y}]e^{-j\beta z}e^{j\delta}$$

Grouping components together, this becomes

$$\mathbf{E}_{\mathbf{s}}^{T} = E_0[(1 + e^{j\delta})\mathbf{a_x} - j(1 - e^{j\delta})\mathbf{a_y}]e^{-j\beta z}$$

Factoring out an overall phase term, $e^{j\delta/2}$, we obtain

$$\mathbf{E}_{\mathbf{s}}^{T} = E_0 e^{j\delta/2}\big[(e^{-j\delta/2} + e^{j\delta/2})\mathbf{a_x} - j(e^{-j\delta/2} - e^{j\delta/2})\mathbf{a_y}\big]e^{-j\beta z}$$

From Euler's identity, we find that $e^{j\delta/2} + e^{-j\delta/2} = 2\cos\delta/2$, and $e^{j\delta/2} - e^{-j\delta/2} = 2j\sin\delta/2$. Using these relations, we obtain

$$\mathbf{E}_{\mathrm{s}}^{T} = 2E_0[\cos(\delta/2)\mathbf{a_x} + \sin(\delta/2)\mathbf{a_y}]e^{-j(\beta z - \delta/2)} \tag{82}$$

We recognize (82) as the electric field of a *linearly polarized* wave, whose field vector is oriented at angle $\delta/2$ from the x axis.

The above example shows that any linearly polarized wave can be expressed as the sum of two circularly polarized waves of opposite handedness, and where the linear polarization direction is determined by the relative phase difference between the two waves. Such a representation is convenient (and necessary) when considering, for example, the propagation of linearly polarized light through media which contain organic molecules. These often exhibit spiral structures having left- or right-handed pitch, and will thus interact differently with left- or right-hand circular polarization. As a result, the left circular component could propagate at a different speed than the right circular component, and so the two waves will accumulate a phase difference as they propagate. The direction of the linearly polarized field vector at the output of the material will thus differ from the direction that it had at the input. The extent of this rotation can be used as a measurement tool to aid in material studies.

Polarization issues will become extremely important when we consider wave reflection in the next chapter.

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4. International Telephone and Telegraph Co., Inc.: "Reference Data for Radio Engineers," 7th ed., Howard W. Sams & Co., Indianapolis, Ind., 1985. This handbook has some excellent data on the properties of dielectric and insulating materials.

PROBLEMS

11.1 Show that $E_{xs} = Ae^{j(k_0 z+\phi)}$ is a solution of the vector Helmholtz equation, Sec. 11.1, Eq. (16), for $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and any ϕ and A.

11.2 Let $E(z, t) = 200 \sin 0.2z \cos 10^8 t\mathbf{a_x} + 500 \cos(0.2z + 50^\circ) \sin 10^8 t\mathbf{a_y}$ V/m. Find: (a) $\mathbf{E}$ at $P(0, 2, 0.6)$ at $t = 25$ ns; (b) $|E|$ at P at $t = 20$ ns; (c) E_s; (d) E_s at P.

11.3 An $\mathbf{H}$ field in free space is given as $\mathbf{H}(x, t) = 10 \cos(10^8 t - \beta x)\mathbf{a_y}$ A/m. Find: (a) β; (b) λ; (c) $\mathbf{E}(x, t)$ at $P(0.1, 0.2, 0.3)$ at $t = 1$ ns.

11.4 In phasor form, the electric field intensity of a uniform plane wave in free space is expressed as $\mathbf{E_s} = (40 - j30)e^{-j20z}\mathbf{a_x}$ V/m. Find: (a) ω; (b) β; (c) f; (d) λ; (e) $\mathbf{H_s}$; (f) $\mathbf{H}(z, t)$ at $P(6, -1, 0.07)$, $t = 71$ ps.

11.5 A 150-MHz uniform plane wave in free space is described by $\mathbf{H_s} = (4 + j10)(2\mathbf{a_x} + j\mathbf{a_y})e^{-j\beta z}$ A/m. (a) Find numerical values for ω, λ, and β. (b) Find $\mathbf{H}(z, t)$ at $t = 1.5$ ns, $z = 20$ cm. (c) What is $|E|_{max}$?

11.6 Let $\mu_R = \epsilon_R = 1$ for the field $\mathbf{E}(z, t) = (25\mathbf{a_x} - 30\mathbf{a_y}) \cos(\omega t - 50z)$ V/m. (a) Find ω. (b) Determine the displacement current density, $\mathbf{J_d}(z, t)$. (c) Find the total magnetic flux Φ passing through the rectangle defined by $0 < x < 1$, $y = 0$, $0 < z < 1$, at $t = 0$.

11.7 The phasor magnetic field intensity for a 400-MHz uniform plane wave propagating in a certain lossless material is $(2\mathbf{a_y} - j5\mathbf{a_z})e^{-j25x}$ A/m. Knowing that the maximum amplitude of $\mathbf{E}$ is 1500 V/m, find β, η, λ, v_p, ϵ_R, μ_R, and $\mathbf{H}(x, y, z, t)$.

11.8 Let the fields, $\mathbf{E}(z, t) = 1800 \cos(10^7\pi t - \beta z)\mathbf{a_x}$ V/m and $\mathbf{H}(z, t) = 3.8 \cos(10^7\pi t - \beta z)\mathbf{a_y}$ A/m, represent a uniform plane wave propagating at a velocity of 1.4×10^8 m/s in a perfect dielectric. Find: (a) β; (b) λ; (c) η; (d) μ_R; (e) ϵ_R.

11.9 A certain lossless material has $\mu_R = 4$ and $\epsilon_R = 9$. A 10-MHz uniform plane wave is propagating in the $\mathbf{a_y}$ direction with $E_{x0} = 400$ V/m and $E_{y0} = E_{z0} = 0$ at $P(0.6, 0.6, 0.6)$ at $t = 60$ ns. (a) Find β, λ, v_p, and η. (b) Find $E(t)$. (c) Find $H(t)$.

11.10 Given a 20-MHz uniform plane wave with $\mathbf{H_s} = (6\mathbf{a_x} - j2\mathbf{a_y})e^{-jz}$ A/m, assume propagation in a lossless medium characterized by $\epsilon_R = 5$ and an unknown μ_R. (a) Find λ, v_p, μ_R, and η. (b) Determine $\mathbf{E}$ at the origin at $t = 20$ ns.

11.11 A 2-GHz uniform plane wave has an amplitude $E_{y0} = 1.4$ kV/m at $(0, 0, 0, t = 0)$ and is propagating in the $\mathbf{a_z}$ direction in a medium where $\epsilon'' = 1.6 \times 10^{-11}$ F/m, $\epsilon' = 3.0 \times 10^{-11}$ F/m, and $\mu = 2.5$ μH/m. Find: (a) E_y at $P(0, 0, 1.8$ cm$)$ at 0.2 ns; (b) H_x at P at 0.2 ns.

11.12 The plane wave $\mathbf{E_s} = 300e^{-jkx}\mathbf{a_y}$ V/m is propagating in a material for which $\mu = 2.25$ μH/m, $\epsilon' = 9$ pF/m, and $\epsilon'' = 7.8$ pF/m. If $\omega = 64$ Mrad/s, find: (a) α; (b) β; (c) v_p; (d) λ; (e) η; (f) $\mathbf{H_s}$; (g) $\mathbf{E}(3, 2, 4, 10$ ns$)$.

11.13 Let $jk = 0.2 + j1.5\ \text{m}^{-1}$ and $\eta = 450 + j60\Omega$ for a uniform plane propagating in the $\mathbf{a_z}$ direction. If $\omega = 300$ Mrad/s, find μ, ϵ', and ϵ'' for the medium.

11.14 A certain nonmagnetic material has the material constants $\epsilon'_R = 2$ and $\epsilon''/\epsilon' = 4 \times 10^{-4}$ at $\omega = 1.5$ Grad/s. Find the distance a uniform plane wave can propagate through the material before: (a) it is attenuated by 1 Np; (b) the power level is reduced by one-half; (c) the phase shifts 360°.

11.15 A 10-GHz radar signal may be represented as a uniform plane wave in a sufficiently small region. Calculate the wavelength in centimeters and the attenuation in nepers per meter if the wave is propagating in a nonmagnetic material for which: (a) $\epsilon'_R = 1$ and $\epsilon''_R = 0$; (b) $\epsilon'_R = 1.04$ and $\epsilon''_R = 9.00 \times 10^{-4}$; (c) $\epsilon'_R = 2.5$ and $\epsilon''_R = 7.2$.

11.16 The power factor of a capacitor is defined as the cosine of the impedance phase angle, and its Q is ωCR, where R is the parallel resistance. Assume an idealized parallel-plate capacitor having a dielectric characterized by σ, ϵ', and μ_R. Find both the power factor and Q in terms of the loss tangent.

11.17 Let $\eta = 250 + j30\Omega$ and $jk = 0.2 + j2\text{m}^{-1}$ for a uniform plane wave propagating in the $\mathbf{a_z}$ direction in a dielectric having some finite conductivity. If $|E_s| = 400$ V/m at $z = 0$, find: (a) $\mathcal{P}_{z,av}$ at $z = 0$ and $z = 60$ cm; (b) the average ohmic power dissipation in watts per cubic meter at $z = 60$ cm.

11.18 (a) Find $\mathcal{P}(\mathbf{r}, t)$ if $E_s = 400e^{-j2x}\mathbf{a_y}$ V/m in free space. (b) Find $\mathcal{P}$ at $t = 0$ for $\mathbf{r} = (a, 5, 10)$, where $a = 0, 1, 2,$ and 3. (c) Find $\mathcal{P}$ at the origin for $t = 0, 0.2T, 0.4T,$ and $0.6T$, where T is the oscillation period.

11.19 Perfectly conducting cylinders with radii of 8 mm and 20 mm are coaxial. The region between the cylinders is filled with a perfect dielectric for which $\epsilon = 10^{-9}/4\pi$ F/m and $\mu_R = 1$. If $\mathbf{E}$ in this region is $(500/\rho)\cos(\omega t - 4z)\mathbf{a}_\rho$ V/m, find: (a) ω, with the help of Maxwell's equations in cylindrical coordinates; (b) $\mathbf{H}(\rho, z, t)$; (c) $\mathcal{P}(\rho, z, t)$; (d) the average power passing through every cross section $8 < \rho < 20$ mm, $0 < \phi < 2\pi$.

11.20 If $\mathbf{E_s} = 60\frac{\sin\theta}{r}e^{-j2r}\mathbf{a}_\theta$ V/m and $\mathbf{H_s} = \frac{\sin\theta}{4\pi r}e^{-j2r}\mathbf{a}_\phi$ A/m in free space, find the average power passing outward through the surface $r = 10^6$, $0 < \theta < \pi/3$, $0 < \phi < 2\pi$.

11.21 The cylindrical shell, 1 cm $< \rho <$ 1.2 cm, is composed of a conducting material for which $\sigma = 10^6$ S/m. The external and internal regions are nonconducting. Let $H_\phi = 2000$ A/m at $\rho = 1.2$ cm. (a) Find $\mathbf{H}$ everywhere. (b) Find $\mathbf{E}$ everywhere. (c) Find $\mathcal{P}$ everywhere.

11.22 The inner and outer dimensions of a coaxial copper transmission line are 2 and 7 mm, respectively. Both conductors have thicknesses much greater than δ. The dielectric is lossless and the operating frequency is 400 MHz. Calculate the resistance per meter length of the: (a) inner conductor; (b) outer conductor; (c) transmission line.

11.23 A hollow tubular conductor is constructed from a type of brass having a conductivity of 1.2×10^7 S/m. The inner and outer radii are 9 and 10 mm, respectively. Calculate the resistance per meter length at a frequency of: (a) dc; (b) 20 MHz; (c) 2 GHz.

11.24 (a) Most microwave ovens operate at 2.45 GHz. Assume that $\sigma = 1.2 \times 10^6$ S/m and $\mu_R = 500$ for the stainless steel interior, and find the depth of penetration. (b) Let $E_s = 50\angle 0^\circ$ V/m at the surface of the conductor, and plot a curve of the amplitude of E_s vs. the angle of E_s as the field propagates into the stainless steel.

11.25 A good conductor is planar in form, and it carries a uniform plane wave that has a wavelength of 0.3 mm and a velocity of 3×10^5 m/s. Assuming the conductor is nonmagnetic, determine the frequency and the conductivity.

11.26 The dimensions of a certain coaxial transmission line are $a = 0.8$ mm and $b = 4$ mm. The outer conductor thickness is 0.6 mm, and all condcutors have $\sigma = 1.6 \times 10^7$ S/m. (a) Find R, the resistance per unit length at an operating frequency of 2.4 GHz. (b) Use information from Secs. 5.10 and 9.10 to find C and L, the capacitance and inductance per unit length, respectively. The coax is air-filled. (c) Find α and β if $\alpha + j\beta = \sqrt{j\omega C(R + j\omega L)}$.

11.27 The planar surface $z = 0$ is a brass-Teflon interface. Use data available in Appendix C to evaluate the following ratios for a uniform plane wave having $\omega = 4 \times 10^{10}$ rad/s: (a) $\alpha_{Tef}/\alpha_{brass}$; (b) $\lambda_{Tef}/\lambda_{brass}$; (c) v_{Tef}/v_{brass}.

11.28 A uniform plane wave in free space has electric field vector given by $\mathbf{E}_s = 10e^{-j\beta x}\mathbf{a}_z + 15e^{-j\beta x}\mathbf{a}_y$ V/m. (a) Describe the wave polarization: (b) Find $\mathbf{H}_s$; (c) determine the average power density in the wave in W/m^2.

11.29 Consider a left-circularly polarized wave in free space that propagates in the forward z direction. The electric field is given by the appropriate form of Eq. (80). (a) Determine the magnetic field phasor, $\mathbf{H}_s$; (b) determine an expression for the average power density in the wave in W/m^2 by direct application of Eq. (57).

11.30 The electric field of a uniform plane wave in free space is given by $\mathbf{E}_s = 10(\mathbf{a}_y + j\mathbf{a}_z)e^{-j25x}$. (a) Determine the frequency, f; (b) find the magnetic field phasor, $\mathbf{H}_s$; (c) describe the polarization of the wave.

11.31 A linearly polarized uniform plane wave, propagating in the forward z direction, is input to a lossless *anisotropic* material, in which the dielectric constant encountered by waves polarized along $y(\epsilon_{Ry})$ differs from that seen by waves polarized along $x(\epsilon_{Rx})$. Suppose $\epsilon_{Rx} = 2.15$, $\epsilon_{Ry} = 2.10$, and the wave electric field at input is polarized at 45° to the positive x and y axes. (a) Determine the shortest length of the material such that the wave as it emerges from the output end is circularly polarized; (b) will the output wave be right- or left-circularly polarized?

11.32 Suppose that the length of the medium of Problem 11.31. is made to be *twice* that as determined in the problem. Describe the polarization of the output wave in this case.

11.33 Given a wave for which $\mathbf{E}_s = 15e^{-j\beta z}\mathbf{a}_x + 18e^{-j\beta z}e^{j\phi}\mathbf{a}_y$ V/m: (a) Find $\mathbf{H}_s$; (b) determine the average power density in W/m^2

11.34 Given a general elliptically polarized wave as per Eq. (73):

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{j\phi}\mathbf{a}_y]e^{-j\beta z}$$

(a) Show, using methods similar to those of Example 11.7, that a linearly polarized wave results when superimposing the given field and a phase-shifted field of the form:

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{-j\phi}\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$$

where δ is a constant; (b) Find δ in terms of ϕ such that the resultant wave is linearly polarized along x.