* Free Undamped Linear

$$m\ddot{x} + kx = 0$$

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

$$A_1 = x_0 \qquad \omega_n A_2 = \dot{x}_0$$

$$x(t) = A \cos(\omega_n t - \phi)$$

$$A = \sqrt{A_1 + A_2} \qquad \phi_0 = \tan^{-1}\left(\frac{A_2}{A_1}\right)$$

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

$$A_0 = \sqrt{A_1 + A_2} \qquad \phi_0 = \tan^{-1}\left(\frac{A_1}{A_2}\right)$$

* Free Undamped Torsion

$$J_0\ddot{\theta} + k_t\theta = 0$$

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

$$A_1 = \theta_0 \qquad A_2 = \dot{\theta}_0/\omega_n$$

* Damped

$$m\ddot{x} + c\dot{x} + kx = 0$$

Case 1. *Underdamped system*

$$x(t) = e^{-\zeta\omega_n t}\left\{C_1' \cos \sqrt{1-\zeta^2}\,\omega_n t + C_2' \sin \sqrt{1-\zeta^2}\,\omega_n t\right\}$$

$$C_1' = x_0 \qquad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$x(t) = Xe^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi\right)$$

$$X = \sqrt{(C_1')^2 + (C_2')^2} \qquad \phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right)$$

$$x(t) = X_0 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_0\right)$$

$$X_0 = \sqrt{(C_1')^2 + (C_2')^2} \qquad \phi_0 = \tan^{-1}\left(\frac{C_1'}{C_2'}\right)$$

Case 2. *Critically damped system*

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

$$C_1 = x_0 \qquad C_2 = \dot{x}_0 + \omega_n x_0$$

Case 3. *Overdamped system*

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2-1})\omega_n t}$$

$$C_1 = \frac{x_0\omega_n(\zeta + \sqrt{\zeta^2-1}) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}} \qquad C_2 = \frac{-x_0\omega_n(\zeta - \sqrt{\zeta^2-1}) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}}$$

* Logarithmic Decrement

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)}$$

$$= \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = e^{\zeta\omega_n\tau_d}$$

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$= \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m}$$

$$\frac{x_1}{x_{m+1}} = (e^{\zeta\omega_n\tau_d})^m = e^{m\zeta\omega_n\tau_d}$$

$$\delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right)$$

* Coulomb Damping

$$m\ddot{x} + kx = -\mu N$$

Case 1.

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu N}{k}$$

$$-A_1 = -x_0 + \frac{3\mu N}{k} \qquad A_2 = 0$$

Case 2.

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu N}{k}$$

$$A_3 = x_0 - \frac{\mu N}{k} \qquad A_4 = 0$$

$$x_0 - r\frac{2\mu N}{k} \le \frac{\mu N}{k} \qquad \theta_r = \theta_0 - r\frac{2T}{k_t}$$

$$r \ge \left\{\frac{x_0 - \frac{\mu N}{k}}{\frac{2\mu N}{k}}\right\} \qquad r \ge \left\{\frac{\theta_0 - \frac{T}{k_t}}{\frac{2T}{k_t}}\right\}$$

$$X_m = X_{m-1} - \frac{4\mu N}{k}$$

* Free Vibration with Hysteretic Damping

$$F = kx + c\dot{x}$$

$$F(t) = kx \pm c\omega\sqrt{X^2 - x^2}$$

$$\Delta W = \pi h X^2$$

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) \simeq \ln(1 + \pi\beta) \simeq \pi\beta$$

Dimension less $\beta = \frac{h}{k}$

if($0 < \zeta < 0.3$) $\delta \simeq 2\pi\zeta_{eq}$

$$\zeta_{eq} = \frac{\beta}{2} = \frac{h}{2k}$$

$$c_{eq} = c_c \cdot \zeta_{eq} = 2\sqrt{mk} \cdot \frac{\beta}{2}$$

$$= \beta\sqrt{mk} = \frac{\beta k}{\omega} = \frac{h}{\omega}$$

* Damped System”

$$m\ddot{x} + kx = F_0 \cos\omega t$$

if($r \neq 1$)

$$x(t) = C_1 \cos\omega_n t + C_2 \sin\omega_n t + X\cos\omega t$$

$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2} \qquad C_2 = \frac{\dot{x}_0}{\omega_n}$$

magnification factor (M) = $\frac{X}{\delta_{st}}$ $\qquad \delta_{st} = F_0/k$

if($r = 1$)

$$x(t) = \underbrace{x_0}_{C_1} \cos\omega_n t + \underbrace{\frac{\dot{x}_0}{\omega_n}}_{C_2} \sin\omega_n t + \frac{\delta_{st}\omega_n t}{2} \sin\omega_n t$$

* Forced “Harmonic” & Damped

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t$$

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X\cos(\omega_d - \phi)$$

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

$$X_0 = \left[(x_0 - X\cos\phi)^2 + \frac{1}{\omega_d^2}(\zeta\omega_n x_0 + \dot{x}_0 - \zeta\omega_n X\cos\phi - \omega X\sin\phi)^2\right]^{\frac{1}{2}}$$

$$\tan\phi_0 = \frac{\zeta\omega_n x_0 + \dot{x}_0 - \zeta\omega_n X\cos\phi - \omega X\sin\phi}{\omega_d(x_0 - X\cos\phi)}$$

* Base excitation Underdamped

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY\sin\omega t + c\omega Y\cos\omega t$$
$$= A\sin(\omega t - \alpha)$$

$$A = Y\sqrt{k^2 + (c\omega)^2}$$

$$x_p(t) = X\sin(\omega t - \phi)$$

$$\frac{X}{Y} = \left[\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}\right]^{1/2} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$\phi = \tan^{-1}\left[\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2}\right] = \tan^{-1}\left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right]$$

* Rotating Unbalance Damped

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin\omega t$$

$$X = \frac{me\omega^2}{[(k - M\omega^2)^2 + (c\omega)^2]^{1/2}} \qquad \frac{MX}{me} = \frac{r^2}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}}$$

$$x_p(t) = X\sin(\omega t - \phi)$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - M\omega^2}\right) \qquad \phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

* Damped

***Response Under a General Periodic Force**

$$F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t$$

$$a_j = \frac{2}{\tau}\int_0^{\tau} F(t)\cos j\omega t\, dt, \qquad j = 0, 1, 2, \ldots$$

$$b_j = \frac{2}{\tau}\int_0^{\tau} F(t)\sin j\omega t\, dt, \qquad j = 1, 2, \ldots$$

***Response Under a Periodic Force of Irregular Form**

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} F_i$$

$$a_j = \frac{2}{N}\sum_{i=1}^{N} F_i \cos\frac{2j\pi t_i}{\tau}, \qquad j = 1, 2, \ldots$$

$$b_j = \frac{2}{N}\sum_{i=1}^{N} F_i \sin\frac{2j\pi t_i}{\tau}, \qquad j = 1, 2, \ldots$$

***Convolution Integral**

$$\text{Impulse} = F\Delta t = m\dot{x}_2 - m\dot{x}_1$$

Response to an Impulse

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}$$

Thus the initial conditions are given by $x_0 = 0 \qquad \dot{x}_0 = \frac{1}{m}$

$$x(t) = \frac{Fe^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = Fg(t)$$

Response to a General Forcing Condition

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)\, e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\, d\tau$$

Time-Delayed Step Force

$$x(t) = \frac{F_0}{k\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2} - e^{-\zeta\omega_n(t-t_0)}\cos\{\omega_d(t-t_0) - \phi\}\right]$$

***Free-Vibration Analysis of an Undamped System**

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$

$$\boxed{\begin{aligned} x_1(t) &= X_1\cos(\omega t + \phi) \longrightarrow \dot{x}_1(t) = -\omega X_1 \sin(\omega t + \phi) \longrightarrow \ddot{x}_1(t) = -\omega^2 X_1\cos(\omega t + \phi) \\ x_2(t) &= X_2\cos(\omega t + \phi) \longrightarrow \dot{x}_2(t) = -\omega X_2 \sin(\omega t + \phi) \longrightarrow \ddot{x}_2(t) = -\omega^2 X_2\cos(\omega t + \phi) \end{aligned}}$$

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1m_2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1m_2}\right\}^2 - 4\left\{\frac{(k_1+k_2)(k_2+k_3) - k_2^2}{m_1m_2}\right\}\right]^{1/2}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} \qquad r_2 = \frac{X_2^{(2)}}{X_1^{(2)}}$$

***Coordinate Coupling and Principal Coordinates**

uncoupled $\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -(k_1l_1 - k_2l_2) \\ -(k_1l_1 - k_2l_2) & (k_1l_1^2 + k_2l_2^2) \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Coupled $\begin{bmatrix} m & me \\ me & J_p \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & (k_2l_2' - k_1l_1') \\ (-k_1l_1' + k_2l_2') & (k_1l_1'^2 + k_2l_2'^2) \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

*Forced-Vibration Analysis

* Damped

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\begin{bmatrix} (-\omega^2 m_{11} + i\omega c_{11} + k_{11}) & (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) \\ (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) & (-\omega_2 m_{22} + i\omega c_{22} + k_{22}) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

$$[Z(i\omega)]\vec{X} = \vec{F}_0 \qquad \vec{X} = [Z(i\omega)]^{-1}\vec{F}_0$$

$$[Z(i\omega)]^{-1} = \frac{1}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)} \begin{bmatrix} Z_{22}(i\omega) & -Z_{12}(i\omega) \\ -Z_{12}(i\omega) & Z_{11}(i\omega) \end{bmatrix} \qquad X_2(i\omega) = \frac{-Z_{12}(i\omega)F_{10} + Z_{11}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$[k_e] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \qquad [k_e] = \frac{EI_z}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

*Potential and Kinetic Energy Expressions in Matrix Form

$$V_i = \frac{1}{2}F_i x_i \qquad V = \frac{1}{2}\vec{x}^T[k]\vec{x} \qquad T = \frac{1}{2}\dot{\vec{x}}^T[m]\dot{\vec{x}} \qquad T = \frac{1}{2}\dot{\vec{q}}^T[m]\dot{\vec{q}}$$

*Lagrange s Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j^{(n)}, \qquad j = 1, 2, \dots, n$$

*Eigenvalue Problem

Characteristic Equation

$$[\lambda[k] - [m]]\vec{X} = \vec{0} \qquad \lambda = \frac{1}{\omega^2} \qquad \lambda[I]\vec{X} = [D]\vec{X} \qquad [D] = [k]^{-1}[m]$$

Orthogonality of Normal Modes

$$\vec{X}^{(i)T}[m]\vec{X}^{(i)} = 1$$

*Unrestrained Systems

$$T = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2) = \frac{1}{2}\dot{\vec{x}}^T[m]\dot{\vec{x}}$$

$$V = \frac{1}{2}\{k_1(x_2 - x_1)^2 + k_2(x_3 - x_2)^2\} = \frac{1}{2}\vec{x}^T[k]\vec{x}$$

*Forced Vibration of Undamped Systems Using Modal Analysis

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F}$$

$$\omega^2[m]\vec{X} = [k]\vec{X}$$

$$\vec{x}(t) = [X]\vec{q}(t)$$

$$q_i(t) = q_i(0)\cos\omega_i t + \left(\frac{\dot{q}(0)}{\omega_i}\right)\sin\omega_i t + \frac{1}{\omega_i}\int_0^t Q_i(\tau)\sin\omega_i(t-\tau)\,d\tau, \qquad i = 1, 2, \dots, n$$

*Forced Vibration of Viscously Damped Systems

$$R = \frac{1}{2}\dot{\vec{x}}^T[c]\dot{\vec{x}} \qquad \ddot{q}_i(t) + 2\zeta_i\omega_i\dot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t), \qquad i = 1, 2, \dots, n$$

$$[c] = \alpha[m] + \beta[k]$$

$$\alpha + \omega_i^2\beta = 2\zeta_i\omega_i$$

$$q_i(t) = e^{-\zeta_i\omega_i t}\left\{\cos\omega_{di}t + \frac{\zeta_i}{\sqrt{1-\zeta_i^2}}\sin\omega_{di}t\right\}q_i(0) + \left\{\frac{1}{\omega_{di}}e^{-\zeta_i\omega_i t}\sin\omega_{di}t\right\}\dot{q}_0(0) + \frac{1}{\omega_{di}}\int_0^t Q_i(\tau)e^{-\zeta_i\omega_i(t-\tau)}\sin\omega_{di}(t-\tau)\,d\tau$$