CHAPTER 5

THE SLOPE – DEFLECTION METHOD

5.1 Force and displacement methods of analysis

In all the methods of elastic structural analysis, two basic conditions must be fulfilled. These are :

(1) Static equilibrium.
(2) Compatibility of deformations.

Two main methods of analysis are possible. In the first, static equilibrium is always satisfied and equations are established to express the second condition; compatibility of deformations in the structure. Since these equations use forces as unknowns, the method is referred to as the *force method*. The method of consistent deformations presented in chapter 2 and the method of the equation of three moments presented in chapter 3 are examples of the force method. In the second method, compatibility of deformations in the structure is always maintained and equations are established to express the first condition; static equilibrium of the structure. Since these equations use displacements as unknowns, the method is referred to as the *displacement method*. The slope-deflection method to be considered in this chapter and the moment distribution method to be considered in the following chapter are examples of the displacement method.

5.2 Degree of freedom

The number of possible joint rotations and independent joint translations in a structure is called the degrees of freedom of the structure.

There are three types of degrees of freedom :

(1) The degree of freedom in rotation

The degree of freedom in rotation of a beam or a frame is equal to the number of possible joint rotations. It follows that if,

s_r = degree of freedom in rotation,

j = number of joints including supports,

f = number of fixed supports,

$$s_r = j - f \qquad \dots \quad 5.1$$

(2) The degree of freedom in translation

The degree of freedom in translation of a frame is equal to the number of independent joint translations which can be given to the frame.

Since in general each joint has two joint translations, the total number of possible joint translations $= 2j$. Since, on the other hand, each fixed or hinged support prevents two of these translations, and each roller support or connecting member prevents one of these translations, the total number of the available translational restraints is $(2f + 2h + r + m)$, where

f = number of fixed supports,

h = number of hinged supports,

r = number of roller supports,

m = number of members.

The degree of freedom in translation, s_t, is thus given by :

$$s_t = 2j - (2f + 2h + r + m) \qquad \ldots \quad 5.2$$

(3) Combined degree of freedom

The combined degree of freedom of a frame, s, is the sum of its degrees of freedom in rotation and translation. Thus,

$$s = s_r + s_t \qquad \ldots \quad 5.3$$

As opposed to the degree of statical indeterminancy, n, which refers to the number of redundant forces, s is sometimes called the degree of *kinematic indeterminancy*. If $s = 0$, the structure is said to be kinematically determinate. The fixed beam is an example of a kinematically determinate structure.

5.3 Outlines of the slope-deflection method

The slope-deflection method is applicable to the analysis of statically indeterminate beams and frames. It is particularly useful for the analysis of highly statically indeterminate structures which generally have a low degree of kinematic indeterminancy.

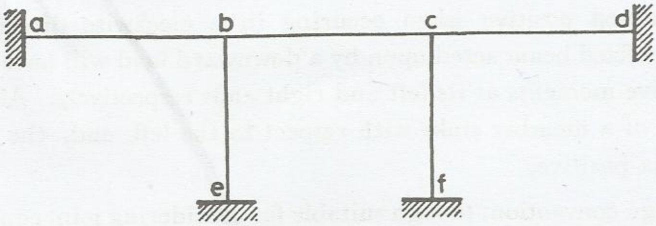


Fig. 5.1

This is demonstrated by the frame shown in Fig. 5.1. The frame is nine times statically indeterminate. This calls for the simultaneous solution of nine equations if the force method of analysis were to be used. On the other hand, the frame has only two unknown rotations, α_b and α_c, and is thus kinematically indeterminate to the second degree. This calls for the simultaneous solution of two equations only if the slope-deflection method is used.

The general procedure for the analysis is outlined below :

(1) The degree of freedom in rotation and translation are found separately. This is done either by inspection or by the application of equations 5.1 and 5.2.

(2) The moments acting on the ends of individual members of the structure are expressed in terms of the rotations of the joints at the ends of the member, the rotation of the member itself (if any) and the load acting on the member. This is done by the slope-deflection equations to be derived in section 5.5.

(3) The conditions of static equilibrium provide as many equations as there are joint and member rotations. When solved simultaneously, these equations give the unknown rotations.

(4) Back-substitution of the thus-determined rotations into the slope-deflection equations yields the moments acting on the ends of individual members.

5.4 Sign conventions

Before proceeding to derive the basic slope-deflection equations, it is necessary to establish specified sign conventions for the moments and rotations.

Moments acting on the ends of members, joint and member rotations are considered positive when occuring in a clockwise direction. For example, a fixed beam acted upon by a downward load will have negative and positive moments at its left and right ends respectively. Also, if the right end of a member sinks with respect to the left end, the member rotation is positive.

This sign convention, though suitable for considering joint equilibrium, contradicts the graphical presentation of the bending moments along the length of a member where moments producing tension in the lower fibers of the member are considered positive and vice versa. Nevertheless, once the end moments are obtained it is quite easy to change over to the usual graphical convention for plotting B.M.Ds.. This is done by drawing free-body diagrams for individual members with the end moments indicated according to the sign convention used in the slope-deflection method.

5.5 Derivation of the slope-deflection equations

Fig. 5.2a shows a single member ab of a continuous beam or a frame. As a result of the loads applied to the structure, the member considered will assume the given general deformed shape. Ends a and b have been caused to rotate through angles α_a and α_b and the axis of the member has rotated an angle $\psi = \Delta/L$ as a result of end b settling an amount Δ

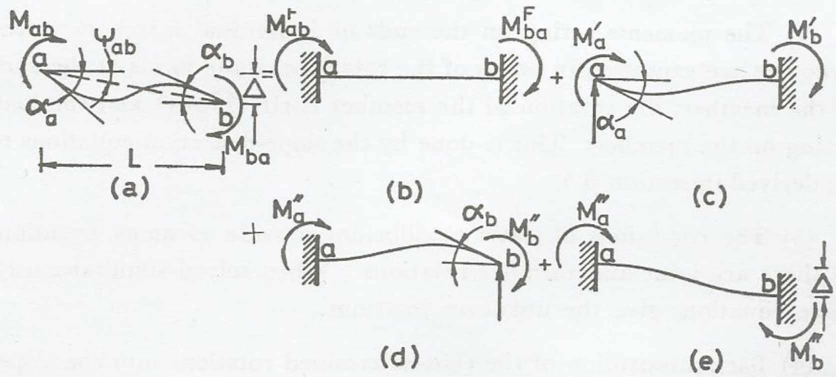


Fig. 5.2

relative to end a. It is now required to find expressions for the final moments M_{ab} and M_{ba} acting on ends a and b in terms of the rotations α_a and α_b at joints a and b, the rotation of the member, ψ_{ab} and the loads acting on the member.

First assume, as in Fig. 5.2b, that ends a and b are fixed, i.e. with zero rotation at the ends. The loads on the member produce the fixed-end moments M^F_{ab} and M^F_{ba} indicated in Fig. 5.2b in their assumed positive direction. These fixed-end moments must be corrected to allow for the end rotations α_a and α_b and the member rotation ψ. The effects of these rotations will be found separately and the final result will then be obtained by superposition.

In Fig. 5.2c the tangent to the elastic curve at a is rotated by a moment M'_a while end b is kept fixed. Using the second moment-area theorem it may be easily shown that $M'_b = \frac{1}{2} M'_a$, and from the first moment-area theorem, $\alpha_a = M'_a L/4EI$. Hence,

$$M'_a = 4\,EI\,\alpha_a/L \quad \text{and} \quad M'_b = 2\,EI\,\alpha_a/L.$$

Similarly, with reference to Fig. 5.2d it may be shown that :

$$M''_a = 2\,EI\,\alpha_b/L \quad \text{and} \quad M''_b = 4\,EI\,\alpha_b/L$$

In Fig. 5.2e, end b is shown displaced relative to end a by an amount Δ while the tangents to the elastic curve at a and b are kept fixed against rotation. The end moments M'''_a and M'''_b may again be expressed in terms of the member rotation ψ using the moment-area theorems. Thus,

$$M'''_a = M'''_b = -6\,EI\,\Delta/L^2 = -6\,EI\,\psi/L.$$

If now, for each end of the member all the end moments corresponding to the various deformations are added to the fixed-end moments, the following expressions are obtained :

$$M_{ab} = M^F_{ab} + 2\,EI/L\,(2\,\alpha_a + \alpha_b - 3\,\psi)$$
$$\qquad \ldots \quad 5.4$$
$$M_{ba} = M^F_{ba} + 2\,EI/L\,(2\,\alpha_b + \alpha_a - 3\,\psi)$$

Equations 5.4 are the basic slope-deflection equations. If there is no relative displacement between the ends of the member, $\psi = 0$ and these equations reduce to :

$$M_{ab} = M^F_{ab} + 2\,EI/L\,(2\,\alpha_a + \alpha_b)$$
$$\qquad \ldots \quad 5.5$$
$$M_{ba} = M^F_{ba} + 2\,EI/L\,(2\,\alpha_b + \alpha_a)$$

It should be noted that the coefficient (2EI/L) appearing in the slope-deflection equations is generally different for each member in the structure.

If the (2EI/L)-values for all the members are made n times smaller (or larger), the effect will only be to make all the rotations n times larger (or smaller) while the products (2 EI/L) $(2\ \alpha_a + \alpha_b - 3\ \psi)$ and (2 EI/L) $(2\ \alpha_b + \alpha_a - 3\ \psi)$ in equations 5.4 or (2 EI/L) $(2\ \alpha_a + \alpha_b)$ and (2 EI/L) $(2\ \alpha_b + \alpha_a)$ in equations 5.5 remain unchanged, and consequently the values of the end moments remain also unchanged. Thus, if the actual values of the rotations are not of interest, as is usually the case, the relative values of (2 EI/L) rather than their absolute values may be used in the slope-deflection equations. If the relative values of (2 EI/L) are denoted by K, the slope-deflection equations in 5.4 and 5.5 may be restated as :

$$M_{ab} = M^F{}_{ab} + K_{ab}\ (2\ \alpha_a + \alpha_b - 3\ \psi)$$

$$M_{ba} = M^F{}_{ba} + K_{ab}\ (2\ \alpha_b + \alpha_a - 3\ \psi) \qquad \ldots\ 5.6$$

and

$$M_{ab} = M^F{}_{ab} + K_{ab}\ (2\ \alpha_a + \alpha_b)$$

$$M_{ba} = M^F{}_{ba} + K_{ab}\ (2\ \alpha_b + \alpha_a) \qquad \ldots\ 5.7$$

5.6 Fixed-end moments

$M^F{}_{ab}$ and $M^F{}_{ba}$ appearing in the slope-deflection equations are the moments that develop at the ends of a member assuming it to be restrained against translation and rotation as in the case of a fixed-end beam.

Expressions for these moments due to common cases of loading are given in Fig. 5.3. Expression for other cases of loading may be found from the first principles or from the given expressions using the principle of superposition. It should be remembered that the given expressions are for prismatic members.

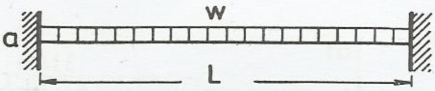
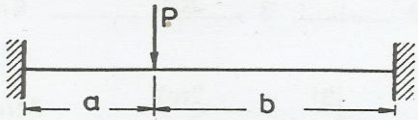
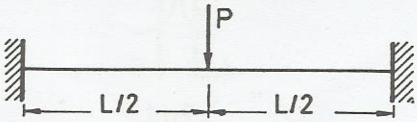
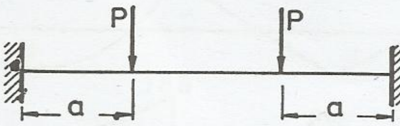
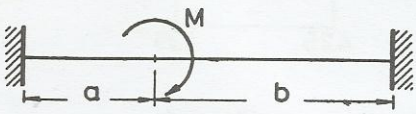
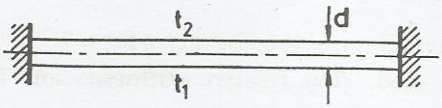
M^F_{ab}	Loading Case	M^F_{ba}
$-\dfrac{wL^2}{12}$		$\dfrac{wL^2}{12}$
$-\dfrac{Pab^2}{L^2}$		$\dfrac{Pba^2}{L^2}$
$-\dfrac{PL}{8}$		$\dfrac{PL}{8}$
$-\dfrac{Pa(L-a)}{L}$		$\dfrac{Pa(L-a)}{L}$
$\dfrac{Mb(2a-b)}{L^2}$		$\dfrac{Ma(2b-a)}{L^2}$
$\dfrac{\alpha_t EI(t_2-t_1)}{d}$		$-\dfrac{\alpha_t EI(t_2-t_1)}{d}$

Fig. 5.3

5.7 Applications to statically indeterminate beams

In this section, the application of the slope-deflection equations to the analysis of statically indeterminate beams will be considered. Beams without joint translation or known joint translation are dealt with.

It should be noticed that only the relative values of K need be known for the analysis of beams having no joint translation. On the other hand, the absolute values of K must appear in the slope-deflection equations for beams with known joint translation.

Example 5.1 Draw the B.M. and S.F.Ds. for the beam shown in Fig. 5.4a if the moment of inertia varies as indicated.

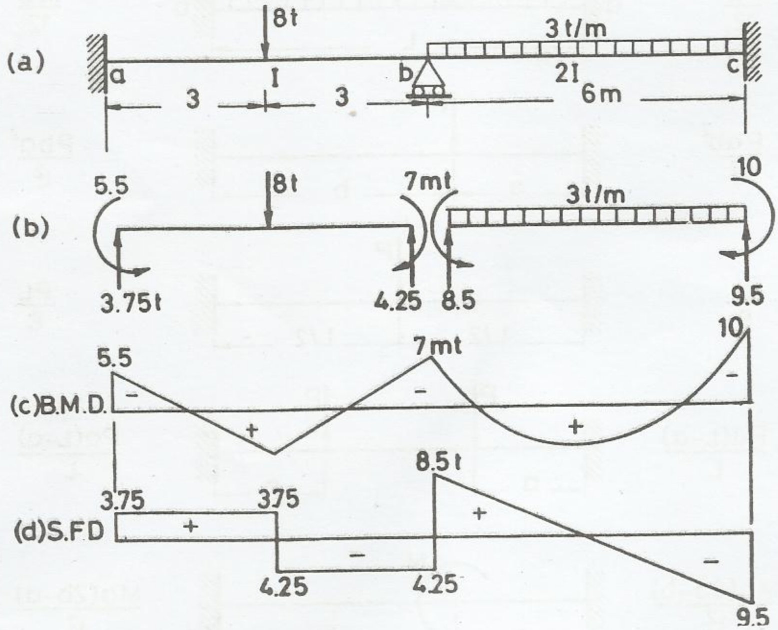


Fig. 5.4

Solution : Since there is no joint translation, form 5.7 of the slope-deflection equations may be used. The relative stiffnesses and fixed-end moments are found first.

$K_{ab} : K_{bc} = I/6 : 2I/6 = 1 : 2$

$M^F{}_{ab} = -8 \times 6/8 = -6$ m.t., $\quad M^F{}_{ba} = 8 \times 6/8 = +6$ m.t.

$M^F{}_{bc} = -3 \times 6^2/12 = -9$ m.t., $M^F{}_{cb} = 3 \times 6^2/12 = +9$ m.t.

The slope-deflection equations are :

$M_{ab} = -6 + 1\,(\alpha_b)$

$M_{ba} = +6 + 1\,(2\,\alpha_b)$

$M_{bc} = -9 + 2\,(2\,\alpha_b)$

$M_{cb} = +9 + 2\,(\alpha_b)$

For the single unknown α_b appearing in the slope-deflection equations, one equation of statics is required. This is provided by the condition of equilibrium at joint b; $\Sigma M_b = 0$. Thus,

$M_{ba} + M_{bc} = 0.$

Substituting from the slope-deflection equations into the joint equilibrium equation,

$$6 + 2\,a_b - 9 + 4\,a_b = 0, \qquad a_b = 1/2$$

By back-substitution in the slope-deflection equations,

$$M_{ab} = -6 + 1\,(1/2) = -5.5 \text{ m.t.}$$
$$M_{ba} = +6 + 1\,(2\times1/2) = +7 \text{ m.t.}$$
$$M_{bc} = -9 + 2\,(2\times1/2) = -7 \text{ m.t.}$$
$$M_{cb} = +9 + 2\,(1/2) = +10 \text{ m.t.}$$

With the end moments thus determined, the free-body diagrams for the two members may be drawn as shown in Fig. 5.4b. The corresponding B.M. and S.F.Ds. are obtained in the usual manner and are as shown in Figs. 5.4c and d respectively.

It should be noted that in this problem there are three unknown moments and one unknown angle. It is obvious therefore that a solution by the displacement method will be simpler than a solution based on the force method.

Example 2.5 Draw the B.M. and S.F.Ds. for the beam shown in Fig. 5.5a if EI = constant.

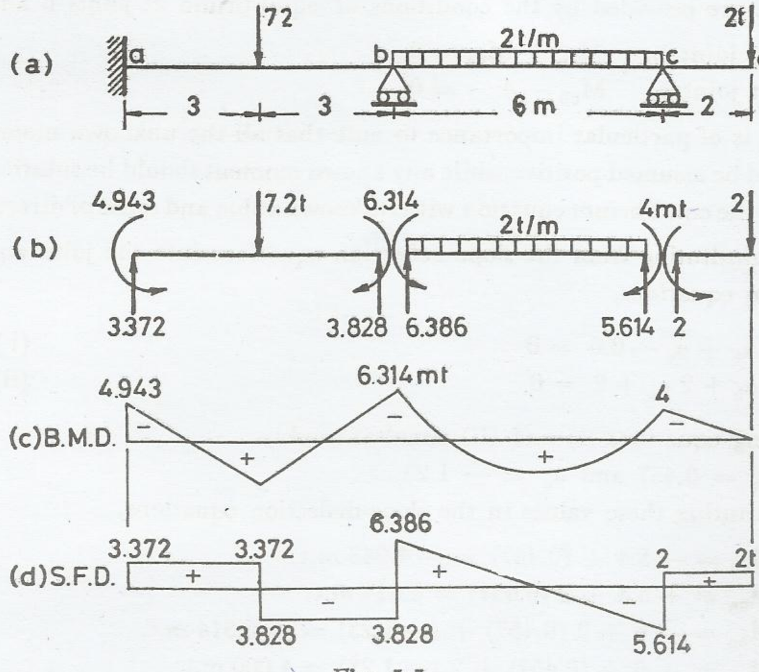


Fig. 5.5

Solution : Since there is no joint translation, form 5.7 of the slope-deflection equations may be used.

The relative stiffnesses and fixed-end moments are as follows :

$$K_{ab} : K_{bc} = I/6 : I/6 = 1 : 1$$

$$M^F{}_{ab} = -PL/8 = -7.2 \times 6/8 = -5.4 \text{ m.t.}, \quad M^F{}_{ba} = +5.4 \text{ m.t.}$$

$$M^F{}_{bc} = -wL^2/12 = -2 \times 6^2/12 = -6 \text{ m.t.}, \quad M^F{}_{cb} = +6 \text{ m.t.}$$

Here α_a is known to be zero and there are two unknown α - values; α_b and α_c. The slope-deflection equations are :

$$M_{ab} = -5.4 + 1\,(2\,\alpha_a + \alpha_b)\,, \quad M_{ab} = -5.4 + \alpha_b$$
$$M_{ba} = +5.4 + 1\,(2\,\alpha_b + \alpha_a)\,, \quad M_{ba} = +5.4 + 2\,\alpha_b$$
$$M_{bc} = -6 \;\; + 1\,(2\,\alpha_b + \alpha_c)\,, \quad M_{bc} = -6 + 2\,\alpha_b + \alpha_c$$
$$M_{cb} = +6 \;\; + 1\,(2\,\alpha_c + \alpha_b)\,, \quad M_{cb} = +6 + \alpha_b + 2\,\alpha_c$$

(Note that a slope-deflection equation needs not be written for cantilever cd as it is statically determinate. Considering a section just to the right of c, $M_{cd} = -2 \times 2 = -4$ m.t. The negative sign for the moment is required since the moment acting on the end of member cd is anticlockwise.

With two unknown rotations, two equations of statics are required. These are provided by the conditions of equilibrium at joints b and c.

At joint b, $\quad M_{ba} + M_{bc} = 0$

At joint c, $\quad M_{cb} - 4 \quad = 0$

It is of particular importance to note that all the unknown moments should be assumed positive while any known moment should be substituted for in the equilibrium equation with its known value and sense of direction.

Substituting from the slope-deflection equations into the joint equilibrium equations,

$$4\,\alpha_b + \alpha_c - 0.6 = 0 \qquad \text{(i)}$$
$$\alpha_b + 2\,\alpha_c + 2 = 0 \qquad \text{(ii)}$$

Solving equations (i) and (ii) simultaneously,

$$\alpha_b = 0.457 \text{ and } \alpha_c = -1.23$$

Substituting these values in the slope-deflection equations,

$$M_{ab} = -5.4 + (0.457) = -4.943 \text{ m.t.}$$
$$M_{ba} = +5.4 + 2\,(0.457) = 6.314 \text{ m.t.}$$
$$M_{bc} = -6 + 2\,(0.457) + (-1.23) = -6.314 \text{ m.t.}$$
$$M_{cb} = +6 + (0.457) + 2\,(-1.23) = 4.000 \text{ m.t.}$$

Having thus determined the end moments, the free-body diagrams for all the members may be drawn as shown in Fig. 5.5b. The corresponding B.M. and S.F.Ds. are obtained in the usual manner and are as shown in Figs. 5.5c and d respectively.

Example 5.3 Analyse the continuous beam shown in Fig. 5.6a if the moment of inertia varies along the beam as indicated.

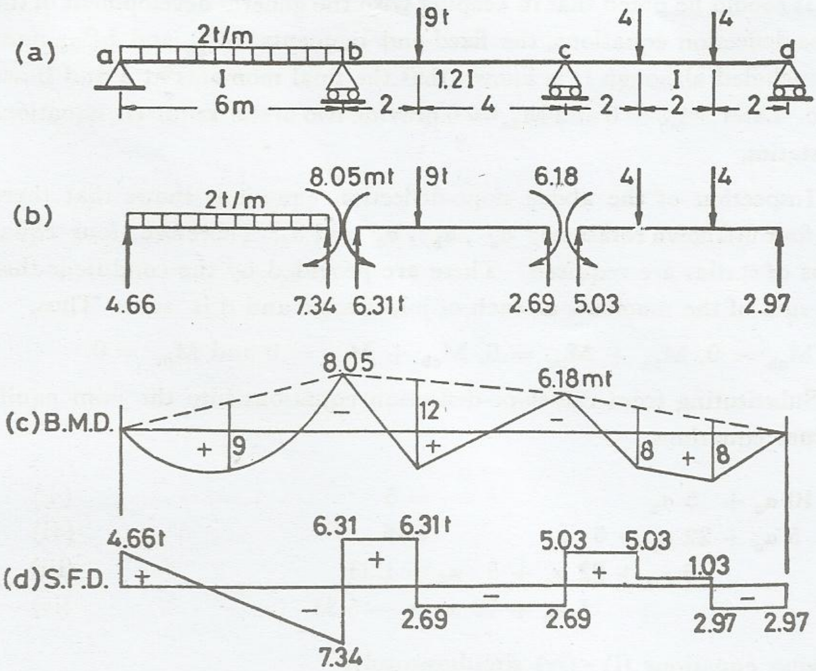


Fig. 5.6

Solution : The relative stiffnesses and fixed-end moments are as follows :

$K_{ab} : K_{bc} : K_{cd} = I/6 : 1.2I/6 : I/6 = 5 : 6 : 5$

$M^F{}_{ab} = -\,wL^2/12 = -\,2 \times 6^2/12 = -\,6$ m.t., $\quad M^F{}_{ba} = +\,6$ m.t

$M^F{}_{bc} = -\,Pab^2/L^2 = -\,9 \times 2 \times 4^2/6^2 = -\,8$ m.t.

$M^F{}_{cb} = Pba^2/L^2 = 9 \times 4 \times 2^2/6^2 = +\,4$ m.t.

$M^F{}_{cd} = -\,Pa\,(L - a)/L = -\,4 \times 2 \times 4/6 = -\,5.33$ m.t.,

$M^F{}_{dc} = +\,5.33$ m.t.

The slope-deflection equations are :

$$M_{ab} = -6 + 5\,(2\,a_a + a_b) \quad , \quad M_{ab} = -6 + 10\,a_a + 5\,a_b$$
$$M_{ba} = +6 + 5\,(2\,a_b + a_a) \quad , \quad M_{ba} = +6 + 5\,a_a + 10\,a_b$$
$$M_{bc} = -8 + 6\,(2\,a_b + a_c) \quad , \quad M_{bc} = -8 + 12\,a_b + 6\,a_c$$
$$M_{cb} = +4 + 6\,(2\,a_c + a_b) \quad , \quad M_{cb} = +4 + 6\,a_b + 12\,a_c$$
$$M_{cd} = -5.33 + 5\,(2\,a_c + a_d) \; , \; M_{cd} = -5.33 + 10\,a_c + 5\,a_d$$
$$M_{dc} = +5.33 + 5\,(2\,a_d + a_c) \; , \; M_{dc} = +5.33 + 5\,a_c + 10\,a_d$$

It should be noted that in keeping with the general development of the slope-deflection equations, the fixed-end moments $M^F{}_{ab}$ and $M^F{}_{dc}$ must be included although it is known that the final moments at a and b are zero. Later, $M_{ab} = 0$ and $M_{dc} = 0$ provide two of the required equations of statics.

Inspection of the above slope-deflection equations shows that there are four unknown rotations; a_a , a_b , a_c and a_d. Therefore, four equations of statics are required. These are provided by the conditions that the sum of the moments at each of joints a,b,c and d is zero. Thus,

$$M_{ab} = 0,\; M_{ba} + M_{bc} = 0,\; M_{cb} + M_{cd} = 0 \text{ and } M_{dc} = 0.$$

Substituting from the slope-deflection equations into the joint equilibrium equations,

$$10\,a_a + 5\,a_b = 6 \qquad \text{(i)}$$
$$5\,a_a + 22\,a_b + 6\,a_c = 6 \qquad \text{(ii)}$$
$$6\,a_b + 22\,a_c + 5\,a_d = 1.33 \qquad \text{(iii)}$$
$$5\,a_c + 10\,a_d = -5.33 \qquad \text{(iv)}$$

Solving equations (i) - (iv) simultaneously,

$$a_a = 0.663, \quad a_b = -0.126, \quad a_c = 0.245 \quad \text{and} \quad a_d = -0.66$$

Substituting these values into the slope-deflection equatiens,

$$M_{ab} = -6 + 10\,(0.663) + 5\,(-0.126) = -6 + 6.63 - 0.63 = 0$$
$$M_{ba} = +6 + 5\,(0.663) + 10\,(-0.126) = +6 + 3.31 - 1.26 = 8.05 \text{ m.t.}$$
$$M_{bc} = -8 + 12\,(-0.126) + 6\,(0.245) = -8 - 1.512 + 1.47$$
$$= -8.05 \text{ m.t.}$$
$$M_{cb} = +4 + 6\,(-0.126) + 12\,(0.245) = +4 - 0.756 + 2.940 = 6.18 \text{ m.t.}$$
$$M_{cd} = -5.33 + 10\,(0.245) + 5\,(-0.66) = -5.33 + 2.45 - 3.30$$
$$= -6.18 \text{ m.t.}$$
$$M_{dc} = +5.33 + 5\,(0.245) + 10\,(-0.66) = +5.33 + 1.23 - 6.6 = 0$$

Having thus determined the end moments, the free-body diagrams for all the members are obtained and the B.M. and S.F.Ds. are constructed in the usual manner. The result is shown in Figs. 5.6 b-d.

It should be noted that in this problem there are two unknown moments only although by the slope-deflection method four unknown angles are involved. It follows that a solution by the force method will be simpler than that worked out above.

Example 5.4 Analyse the beam shown in Fig. 5.7a in the absence of loads for the following simultaneous support movements :

A downward movement of 0.3 cm. in addition to a clockwise rotation of 0.001 rad. at support a, a downward movement of 1.2 cm. at support b, and a downward movement of 0.6 cm. at support c. EI = 5000 m^2t.

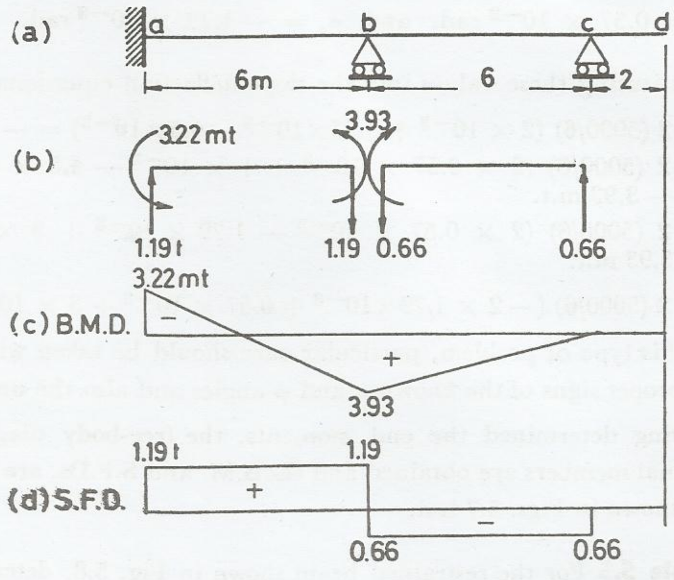


Fig. 5.7

Solution : Since there are finite joint and member rotations, form 5.4 of the slope-deflection equations should be used with the absolute (2EI/L)-values. Further, as there are no loads all the fixed-end moments are zero. In this case it is known that :

$$a_a = +0.001,\ \psi_{ab} = \frac{1.2\text{-}0.3}{600} = 0.0015 \text{ and } \psi_{bc} = -\frac{1.2-0.6}{600} = -0.001$$

The slope-deflection equations are thus as follows :

$$M_{ab} = 2\,(5000/6)\,(2 \times 0.001 + \alpha_b - 3 \times 0.0015)$$
$$M_{ba} = 2\,(5000/6)\,(2\,\alpha_b + 0.001 - 3 \times 0.0015)$$
$$M_{bc} = 2\,(5000/6)\,(2\,\alpha_b + \alpha_c + 3 \times 0.001)$$
$$M_{cb} = 2\,(5000/6)\,(2\,\alpha_c + \alpha_b + 3 \times 0.001)$$

There are two unknown rotations; α_b and α_c. These may be found from the conditions of equilibrium at joints b and c. Thus,

$$M_{ba} + M_{bc} = 0 \quad \text{and} \quad M_{cb} = 0$$

Substituting from the slope-deflection equations into the above two equilibrium equations,

$$4\,\alpha_b + \alpha_c - 0.5 \times 10^{-3} = 0 \qquad \text{(i)}$$
$$\alpha_b + 2\,\alpha_c + 3 \times 10^{-3} = 0 \qquad \text{(ii)}$$

Solving equations (i) and (ii) simultaneously,

$$\alpha_b = 0.57 \times 10^{-3}\ \text{rad. and} \quad \alpha_c = -1.79 \times 10^{-3}\ \text{rad.}$$

Substituting these values into the slope-deflection equations,

$$M_{ab} = 2\,(5000/6)\,(2 \times 10^{-3} + 0.57 \times 10^{-3} - 4.5 \times 10^{-3}) = -3.22\ \text{m.t.}$$
$$M_{ba} = 2\,(5000/6)\,(2 \times 0.57 \times 10^{-3} + 1 \times 10^{-3} - 4.5 \times 10^{-3}) = -3.93\ \text{m.t.}$$
$$M_{bc} = 2\,(5000/6)\,(2 \times 0.57 \times 10^{-3} - 1.79 \times 10^{-3} + 3 \times 10^{-3}) = 3.93\ \text{m.t.}$$
$$M_{cb} = 2\,(5000/6)\,(-2 \times 1.79 \times 10^{-3} + 0.57 \times 10^{-3} + 3 \times 10^{-3}) = 0$$

In this type of problem, particular care should be taken with regard to the proper signs of the known α and ψ angles and also the units used.

Having determined the end moments, the free-body diagrams for individual members are obtained and the B.M. and S.F.Ds. are constructed as shown in Figs. 5.7 b-d.

Example 5.5 For the restrained beam shown in Fig. 5.8, determine the fixing moment at end a if support b settles δ per unit load and EI = constant.

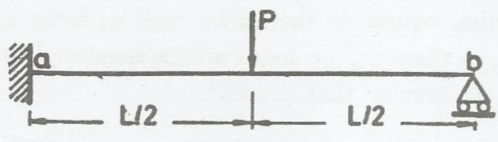


Fig. 5.8

Solution : $\Delta_b = \delta \times Y_b = \delta\,(P/2 + M_{ab}/L)$. Hence, $\psi = \delta\,(P/2 + M_{ab}/L)/L$. Since there is finite rotation of member ab, form 5.4 of the slope-deflection equations should be used. Thus,

$$\begin{aligned} M_{ab} &= -\,PL/8 + (2\,EI/L)\,[\alpha_b - 3\,\delta\,(P/2 + M_{ab}/L)/L] \\ &= -\,PL/8 + 2\,EI\,\alpha_b/L - 6\,EI\,\delta\,(P/2 + M_{ab}/L)/L^2 \end{aligned}$$

$$\begin{aligned} M_{ba} &= +\,PL/8 + (2\,EI/L)\,[2\,\alpha_b - 3\,\delta\,(P/2 + M_{ab}/L)/L] \\ &= +\,PL/8 + 4\,EI\,\alpha_b/L - 6\,EI\,\delta\,(P/2 + M_{ab}/L)/L^2 \end{aligned}$$

There is one unknown rotation, α_b. This is obtained from the condition : $M_{ba} = 0 = PL/8 + 4\,EI\,\alpha_b/L - 6\,EI\,\delta\,(P/2 + Mab/L)/L^2$ or, $2\,EI\,\alpha_b/L = 3\,EI\,\delta\,(P/2 + M_{ab}/L)/L^2 - PL/16$

Substituting this value into the first slope-deflection equation,

$$M_{ab} = -\,(PL/16 + EI\,\delta\,P/2L^2)\,/\,(1/3 + EI\,\delta\,/L^3)$$

If support b does not settle, $\delta = 0$ and $M_{ab} = -\,3PL/16$ which is the known value for a restrained beam subject to a central concentrated load P. It is also noticed that sinking of support b reduces the reaction at b and increases the end moment at a.

5.8 Applications to statically indeterminate frames with no joint translation

Frames that do not involve translation of the joints may be solved using form 5.5. or 5.7 of the slope-deflection equations. In this case, as for statically indeterminate beams, there will be as many joint equilibrium equations as there are unknown joint rotations. After solving these equations, the joint rotations obtained are back-substituted in the slope-deflection equations to determine the moments acting on the ends of the various members and the analysis is then completed in the usual manner.

Translation of the joints is generally prevented as a result of the disposition of the supports and the supporting conditions of the frame or complete symmetry in both frame and loading. Fig. 5.9 shows examples of frames with no joint translation. Joint translation of the frames in Figs. 5.9a and b is prevented due to the supporting conditions. On the other hand, joint translation of the frames in Figs. 5.9c and d, while generally possible, do not occur due to the complete symmetry in frame and loading. It should be noticed that strictly speaking there are slight joint

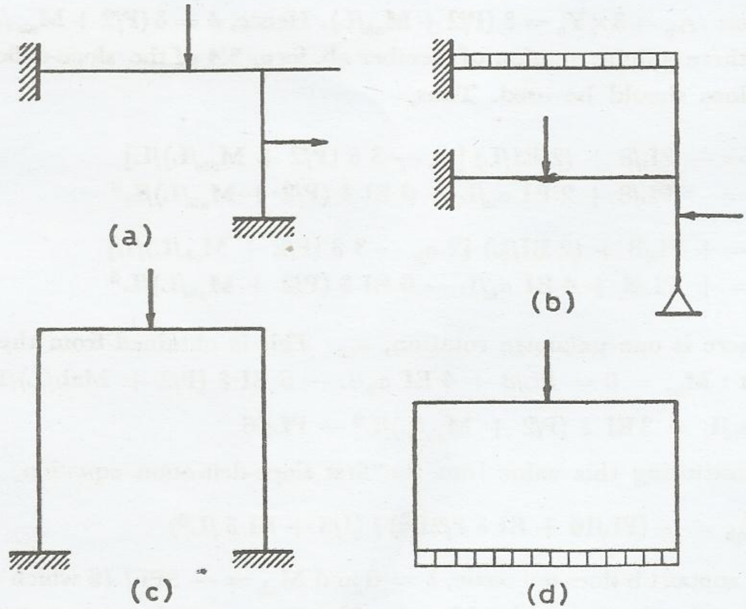


Fig. 5.9

translation due to the axial deformation of the members but this is normally neglected in the analysis of such frames.

Example 5.6 Draw the S.F. and B.M.Ds. for the frame shown in Fig. 5.10a if the moment of inertia varies as indicated.

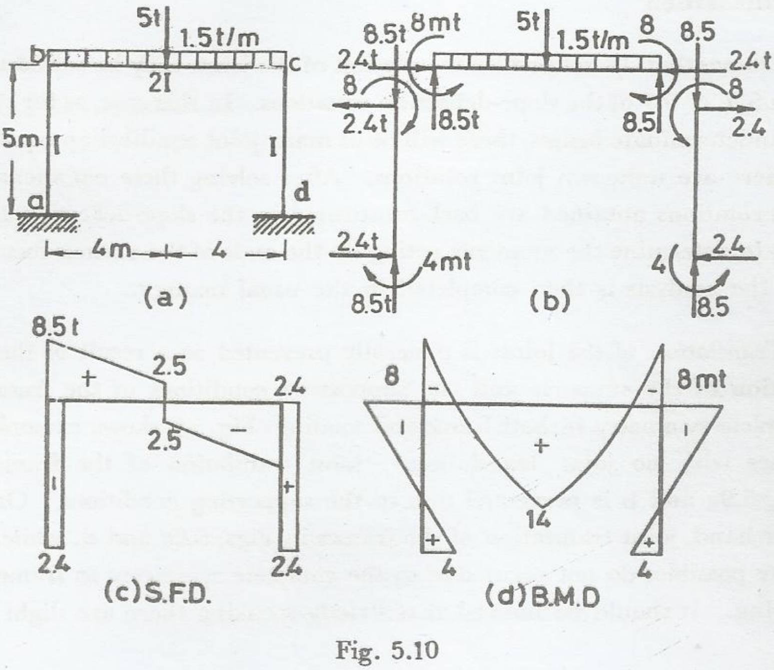


Fig. 5.10

Solution : Since the frame is symmetrical and symmetrically loaded, there will be no joint translation. The relative stiffnesses and fixed-end moments are as follows :

$K_{ab} : K_{bc} : K_{cd} = I/5 : 2I/8 : I/5 = 4 : 5 : 4$

$M^F_{bc} = -(wL^2/12 + PL/8) = -(1.5 \times 8^2/12 + 5 \times 8/8) = -13\,\text{m.t.}$

$M^F_{cb} = +13\,\text{m.t.}$

Here it is known that $\alpha_a = \alpha_b = 0$, and hence the slope-deflection equations are as follows :

$M_{ab} = 0 + 4\,(\alpha_b)$, $M_{ab} = 4\,\alpha_b$
$M_{ba} = 0 + 4\,(2\,\alpha_b)$, $M_{ba} = 8\,\alpha_b$
$M_{bc} = -13 + 5\,(2\,\alpha_b + \alpha_c)$, $M_{bc} = -13 + 10\,\alpha_b + 5\,\alpha_c$
$M_{cb} = +13 + 5\,(2\,\alpha_c + \alpha_b)$, $M_{cb} = +13 + 5\,\alpha_b + 10\,\alpha_c$
$M_{cd} = 0 + 4\,(2\,\alpha_c)$, $M_{cd} = 8\,\alpha_c$
$M_{dc} = 0 + 4\,(\alpha_c)$, $M_{dc} = 4\,\alpha_c$

There are two unknown rotations; α_b and α_c. These may be found from the conditions of equilibrium at joints b and c. Thus,

$M_{ba} + M_{bc} = 0$ and $M_{cb} + M_{cd} = 0$

Substituting from the slope-deflection equations into the two joint equilibrium equations and solving them simultaneously, $\alpha_b = +1$ and $\alpha_b = -1$. The numerically equal values of α_b and α_c are consistent with the symmetry of the frame. (This symmetry could have lead to appreciable simplifications had it been utilized at the outset of the solution. This will be discussed in detail in section 5.11).

Substituting the values of α_b and α_c into the slope-deflection equations,

$M_{ab} = 4\,(1) = 4\,\text{m.t.}$
$M_{ba} = 8\,(1) = 8\,\text{m.t.}$
$M_{bc} = -13 + 10\,(1) + 5\,(-1) = -8\,\text{m.t.}$
$M_{cb} = +13 + 5\,(1) + 10\,(-1) = 8\,\text{m.t.}$
$M_{cd} = 8\,(-1) = -8\,\text{m.t.}$
$M_{dc} = 4\,(-1) = -4\,\text{m.t.}$

The free-body diagram for each member is shown in Fig. 5.10b while the S.F. and B.M.Ds. are shown in Figs. 5.10c and d respectively.

Example 5.7 Analyse the frame shown in Fig. 5.11a if EI = constant.

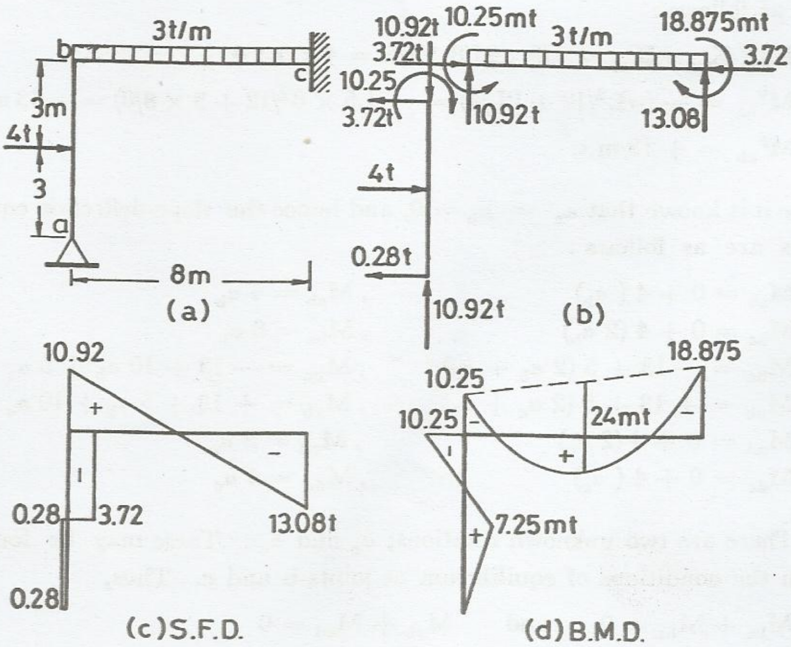


Fig. 5.11

Solution : All the joints in the frame are prevented from translation by virtue of the supporting conditions. The relative stiffnesses and fixed-end moments are as follows :

$K_{ab} : K_{bc} = I/6 : I/8 = 4 : 3$

$M^F_{ab} = -4 \times 6/8 = -3$ m.t. , $M^F_{ba} = +3$ m.t.

$M^F_{bc} = -3 \times 8^2/12 = -16$ m.t. , $M^F_{cb} = +16$ m.t.

Here $\alpha_c = 0$, and the slope-deflection equations are as follows :

$M_{ab} = -3 + 4(2\alpha_a + \alpha_b)$, $M_{ab} = -3 + 8\alpha_a + 4\alpha_b$

$M_{ba} = +3 + 4(2\alpha_b + \alpha_a)$, $M_{ba} = +3 + 4\alpha_a + 8\alpha_b$

$M_{bc} = -16 + 3(2\alpha_b)$, $M_{bc} = -16 + 6\alpha_b$

$M_{cb} = +16 + 3(\alpha_b)$, $M_{cb} = +16 + 3\alpha_b$

The two unknown angles α_a and α_b are found from the two joint equilibrium conditions; $M_{ab} = 0$ and $M_{ba} + M_{bc} = 0$

Substituting from the slope-deflection equations into the joint equilibrium equations and solving simultaneously, $\alpha_a = -5/48$ and $\alpha_b = 23/24$.

Substituting these values back into the slope-deflection equations,

$$M_{ab} = -3 + 8\,(-5/48) + 4\,(23/24) = 0$$

$$M_{ba} = +3 + 4\,(-5/48) + 8\,(23/24) = 10.25 \text{ m.t.}$$

$$M_{bc} = -16 + 6\,(23/24) = -10.25 \text{ m.t.}$$

$$M_{cb} = +16 + 3\,(23/24) = 18.875 \text{ m.t.}$$

The free-body, S.F. and B.M.Ds. are shown in succession in Figs.5.11b,c and d.

Example 5.8 Construct the B.M.D. for the frame shown in Fig. 5.12a if the relative moments of inertia are as indicated.

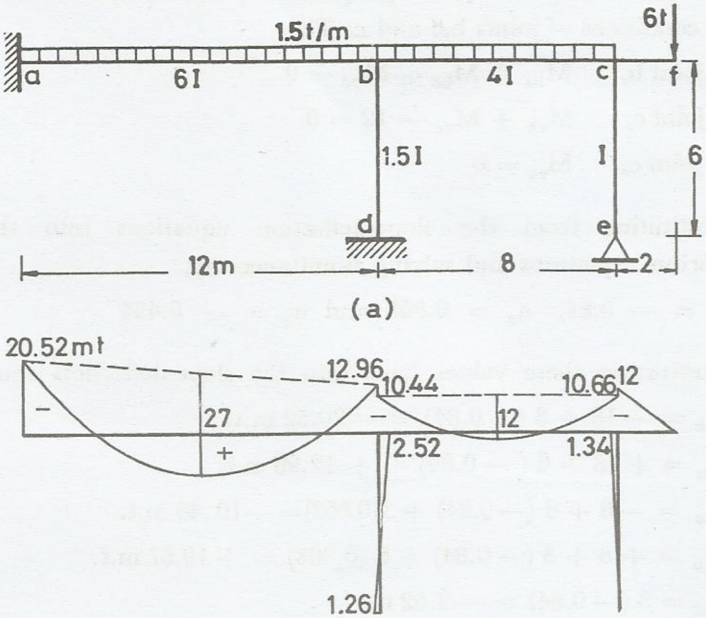


Fig. 5.12

Solution : The relative stiffnesses and fixed -end moments are as follows :

$$K_{ab} : K_{bc} : K_{bd} : K_{ce} = 6I/12 : 4I/8 : 1.5I/6 : I/6 = 3 : 3 : 1.5 : 1$$

$$M^F_{ab} = -1.5 \times 12^2/12 = -18 \text{ m.t.} \qquad , M^F_{ba} = +18 \text{ m.t.}$$

$$M^F_{bc} = -1.5 \times 8^2/12 = -8 \text{ m.t.} \qquad , M^F_{cb} = +8 \text{ m.t.}$$

Here $\alpha_a = \alpha_d = 0$, and the slope-deflection equations are expressed in terms of α_b, α_c and α_e.

$M_{ab} = -18 + 3\,(\alpha_b)$, $M_{ab} = -18 + 3\,\alpha_b$

$M_{ba} = +18 + 3\,(2\,\alpha_b)$, $M_{ba} = +18 + 6\,\alpha_b$

$M_{bc} = -8 + 3\,(2\,\alpha_b + \alpha_c)$, $M_{bc} = -8 + 6\,\alpha_b + 3\,\alpha_c$

$M_{cb} = +8 + 3\,(2\,\alpha_c + \alpha_b)$, $M_{cb} = +8 + 3\,\alpha_b + 6\,\alpha_c$

$M_{bd} = 0 + 1.5\,(2\,\alpha_b)$, $M_{bd} = 3\,\alpha_b$

$M_{db} = 0 + 1.5\,(\alpha_b)$, $M_{db} = 1.5\,\alpha_b$

$M_{ce} = 0 + 1\,(2\,\alpha_c + \alpha_e)$, $M_{ce} = 2\,\alpha_c + \alpha_e$

$M_{ec} = 0 + 1\,(2\,\alpha_e + \alpha_c)$, $M_{ec} = \alpha_c + 2\,\alpha_e$

The three unknown angles α_b, α_c and α_e are found from the equilibrium conditions of joints b,c and e. Thus,

At joint b, $M_{ba} + M_{bc} + M_{bd} = 0$

At joint c, $M_{cb} + M_{ce} - 12 = 0$

At joint e, $M_{ec} = 0$

Substituting from the slope-deflection equations into the joint equilibrium equations and solving simultaneously,

$\alpha_b = -0.84, \quad \alpha_c = 0.868$ and $\alpha_e = -0.434$

Substitutiug these values back into the slope-deflection equations,

$M_{ab} = -18 + 3\,(-0.84) = -20.52$ m.t.

$M_{ba} = +18 + 6\,(-0.84) = +12.96$ m.t.

$M_{bc} = -8 + 6\,(-0.84) + 3(0.868) = -10\ .44$ m.t.

$M_{cb} = +8 + 3\,(-0.84) + 6\,(0.863) = +10.67$ m.t.

$M_{bd} = 3\,(-0.84) = -2.52$ m.t.

$M_{db} = 1.5\,(-0.84) = -1.26$ m.t.

$M_{ce} = 2\,(0.863) - 0.434 = +1.34$ m.t.

$M_{ec} = 0.868 + 2\,(-0.434) = 0$

The final B.M.D. is thus as shown in Fig. 5.12b.

5.9 Applications to frames with a single degree of freedom in translation

In general, joints in frames translate as well as rotate. Joint translation may occur as a result of applied lateral loads or lack of symmetry in either frame or loading.

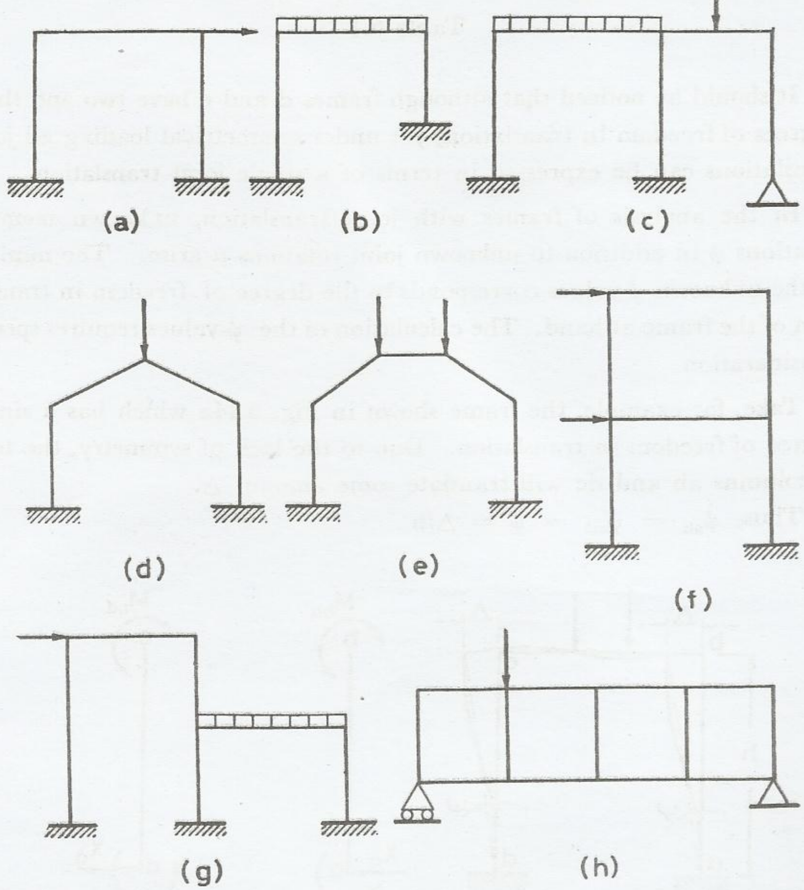


Fig. 5.13

Fig. 5.13 shows several examples of frames with different degrees of freedom in translation. This can be found either by inspection or by the application of equation 5.2 and the result is presented in Table 5.1.

Frame	j	f	h	r	m	$s_t = 2j - (2f + 2h + r + m)$
a	4	2	—	—	3	$2 \times 4 - (2 \times 2 + 3) = 1$
b	4	2	—	—	3	$2 \times 4 - (2 \times 2 + 3) = 1$
c	6	2	1	—	5	$2 \times 6 - (2 \times 2 + 2 \times 1 + 5) = 1$
d	5	2	—	—	4	$2 \times 5 - (2 \times 2 + 4) = 2$
e	6	2	—	—	5	$2 \times 6 - (2 \times 2 + 5) = 3$
f	6	2	—	—	6	$2 \times 6 - (2 \times 2 + 6) = 2$
g	7	3	—	—	6	$2 \times 7 - (2 \times 3 + 6) = 2$
h	10	—	1	1	13	$2 \times 10 - (2 \times 1 + 1 + 13) = 4$

Table 5.1

It should be noticed that although frames d and e have two and three degrees of freedom in translation, yet under symmetrical loading all joint translations can be expressed in terms of a single joint translation.

In the analysis of frames with joint translation, unknown member rotations ψ in addition to unknown joint rotations α arise. The number of the unknown ψ-values corresponds to the degree of freedom in translation of the frame at hand. The calculation of the ψ-values requires special consideration.

Take, for example, the frame shown in Fig. 5.14a which has a single degree of freedom in translation. Due to the lack of symmetry, the tops of columns ab and dc will translate some amount Δ.

Thus, $\psi_{ab} = \psi_{cd} = \psi = \Delta/h$

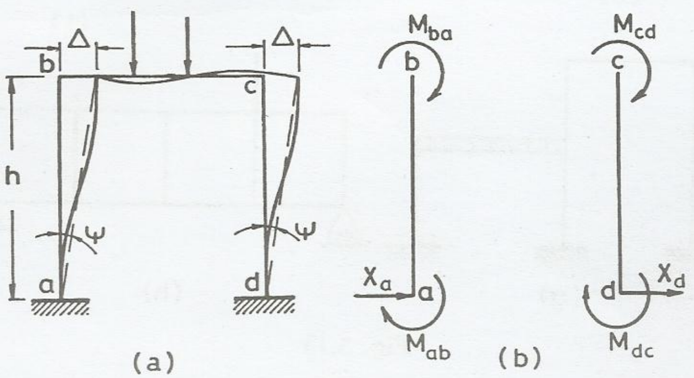


Fig. 5.14

Of course at the beginning of the solution, the magnitude of the translation Δ, and hence the member rotation ψ, is not known.

Referring to Fig. 5.14b, the moments acting on the ends of the columns are shown in their assumed positive direction. From statics,

$$X_a = (M_{ab} + M_{ba})/h \quad \text{and} \quad X_d = (M_{dc} + M_{cd})/h$$

and since there are no horizontal loads acting on the frame, the sum of these reaction components must be zero. Thus,

$$(M_{ab} + M_{ba})/h + (M_{dc} + M_{cd})/h = 0$$

This equation is commonly known as the *shear equation*. There will always be as many shear equations as there are unknown member rotations. The student should note in particular that in the shear equations, as in the joint equilibrium equations, all unknown moments should be assumed positive.

The simultaneous solution of the shear equation and the other two equations provided by the conditions of equilibrium at joints b and c; $M_{ba} + M_{bc} = 0$ and $M_{cb} + M_{cd} = 0$, yields the unknown rotations α_a, α_b and ψ.

Once these rotations are determined, the moments acting on the ends of the various members may be found by back-substitution in the slope-deflection equations and the analysis completed in the usual manner.

Example 5.9 Calculate the reactions of the frame shown in Fig. 5.15a if the relative moments of inertia are as indicated.

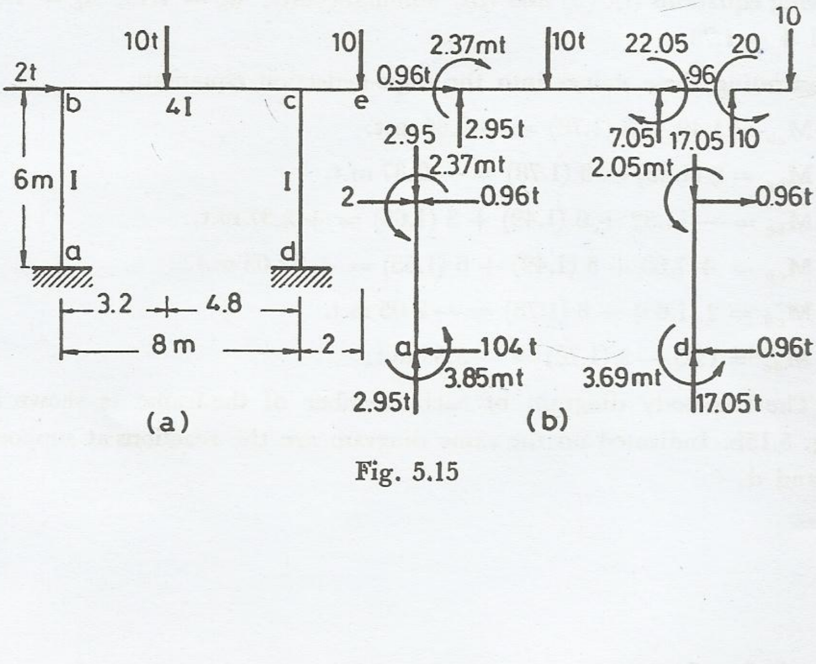


Fig. 5.15

Solution : There are three unknown rotations; two joint rotations α_b and α_c and one member rotation $\psi\ (= \psi_{ab} = \psi_{cd})$. The relative stiffnesses and fixed-end moments are as follows :

$$K_{ab} : K_{bc} : K_{cd} = I/6 : 4I/8 : I/6 = 1 : 3 : 1$$
$$M^F_{bc} = -10 \times 3.2 \times 4.8^2/8^2 = -11.52 \text{ m.t.}$$
$$M^F_{cb} = +10 \times 4.8 \times 3.2^2/8^2 = +7.68 \text{ m.t.}$$

The slope-deflection equations are as follows :

$$M_{ab} = 0 + 1\,(\alpha_b - 3\,\psi) \qquad , M_{ab} = \alpha_b - 3\psi$$
$$M_{ba} = 0 + 1\,(2\,\alpha_b - 3\,\psi) \qquad , M_{ba} = 2\,\alpha_b - 3\psi$$
$$M_{bc} = -11.52 + 3\,(2\,\alpha_b + \alpha_c) \quad , M_{bc} = -11.52 + 6\,\alpha_b + 3\,\alpha_c$$
$$M_{cb} = +7.68 + 3\,(2\,\alpha_c + \alpha_b) \quad , M_{cb} = +7.68 + 3\,\alpha_b + 6\,\alpha_c$$
$$M_{cd} = 0 + 1\,(2\,\alpha_c - 3\,\psi) \qquad , M_{cd} = 2\,\alpha_c - 3\psi$$
$$M_{dc} = 0 + 1\,(\alpha_c - 3\,\psi) \qquad , M_{dc} = \alpha_c - 3\psi$$

At joint b, $M_{ba} + M_{bc} = 0$

At joint c, $M_{cb} + M_{cd} - 20 = 0$

Shear equation, $(M_{ab} + M_{ba})/6 + (M_{cd} + M_{dc})/6 + 2 = 0$

Substituting from the slope-deflection equations into the three equations of statics,

$$8\,\alpha_b + 3\,\alpha_c - 3\,\psi = 11.52 \qquad \text{(i)}$$
$$3\,\alpha_b + 8\,\alpha_c - 3\,\psi = -7.68 \qquad \text{(ii)}$$
$$-3\,\alpha_b - 3\,\alpha_c + 12\psi = 12 \qquad \text{(iii)}$$

Solving equations (i), (ii) and (iii) simultaneously, $\alpha_b = 1.49$, $\alpha_c = 1.65$ and $\psi = 1.78$.

Substituting these values into the slope-deflection equations,

$$M_{ab} = 1.49 - 3\,(1.78) = -3.85 \text{ m.t.}$$
$$M_{ba} = 2\,(1.49) - 3\,(1.78) = -2.37 \text{ m.t.}$$
$$M_{bc} = -11.52 + 6\,(1.49) + 3\,(1.65) = +2.37 \text{ m.t.}$$
$$M_{cb} = +7.68 + 3\,(1.49) + 6\,(1.65) = +22.05 \text{ m.t.}$$
$$M_{cd} = 2\,(1.65) - 3\,(1.78) = -2.05 \text{ m.t.}$$
$$M_{dc} = 1.65 - 3\,(1.78) = -3.69 \text{ m.t.}$$

The free-body diagram of each member of the frame is shown in Fig. 5.15b. Indicated on the same diagram are the reactions at supports a and d.

Example 5.10 Construct the B.M.D. for the frame shown in Fig. 5.16a if EI = constant.

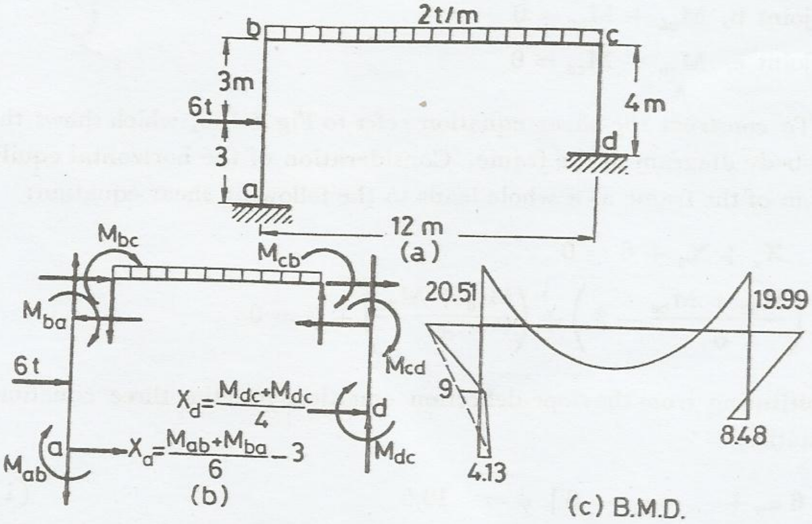


Fig. 5.16

Solution : The relative stiffnesses and fixed-end moments are as follows:

$K_{ab} : K_{bc} : K_{cd} = I/6 : I/12 : I/4 = 2 : 1 : 3$

$M^F{}_{ab} = -6 \times 6/8 = -4.5$ m.t. , $M^F{}_{ba} = +4.5$ m.t.

$M^F{}_{bc} = -2 \times 12^2/12 = -24$m.t. , $M^F{}_{cb} = +24$ m.t.

Neither the frame nor its loading is symmetrical. This lack of symmetry will cause member bc to translate horizontally some amount Δ. Thus,

$$\psi_{ab} = \Delta/6 = \psi \; , \; \psi_{bc} = 0 \text{ and } \psi_{cd} = \Delta/4 = 1.5\psi$$

There are three unknown rotations; two joint rotations α_b and α_c and one member rotation ψ.

The slope-deflection equations are as follows :

$M_{ab} = -4.5 + 2\,(\alpha_b - 3\,\psi)$, $M_{ab} = -4.5 + 2\,\alpha_b - 6\,\psi$

$M_{ba} = +4.5 + 2\,(2\,\alpha_b - 3\,\psi)$, $M_{ba} = +4.5 + 4\,\alpha_b - 6\,\psi$

$M_{bc} = -24 + 1\,(2\,\alpha_b + \alpha_c)$, $M_{bc} = -24 + 2\,\alpha_b + \alpha_c$

$$M_{cb} = +24 + 1\,(2\,\alpha_c + \alpha_b) \qquad , M_{cb} = +24 + \alpha_b + 2\,\alpha_c$$

$$M_{cd} = 0 + 3\,(2\,\alpha_c - 3 \times 1.5\,\psi)\,,\ M_{cd} = 6\,\alpha_c - 13.5\,\psi$$

$$M_{dc} = 0 + 3\,(\,\alpha_c - 3 \times 1.5\,\psi)\,,\ M_{dc} = 3\,\alpha_c - 13.5\,\psi$$

At joint b, $M_{ba} + M_{bc} = 0$

At joint c, $M_{cb} + M_{cd} = 0$

To construct the shear equation refer to Fig. 5.16b, which shows the free-body diagram of the frame. Consideration of the horizontal equilibrium of the frame as a whole leads to the following shear equation:

$$X_a + X_d + 6 = 0$$

$$\left(\frac{M_{ab} + M_{ba}}{6} - 3\right) + \left(\frac{M_{cd} + M_{dc}}{4}\right) + 6 = 0$$

Substituting from the slope-deflection equations into the three equations of statics,

$$6\,\alpha_b + \alpha_c - 6\,]\,\psi = 19.5 \qquad \text{(i)}$$

$$\alpha_b + 8\,\alpha_c - 13.5\,\psi = -24 \qquad \text{(ii)}$$

$$-6\,\alpha_b - 13.5\,\alpha_c + 52.5\,\psi = 18 \qquad \text{(iii)}$$

Solving equations (i), (ii) and (iii) simultaneously, $\alpha_b = 3.665$, $\alpha_c = -3.838$ and $\psi = -0.225$

Substituting these values into the slope-deflection equations,

$$M_{ab} = -4.5 + 2\,(3.665) - 6\,(-0.225) = +4.13 \text{ m.t.}$$

$$M_{ba} = +4.5 + 4\,(3.665) - 6\,(-0.225) = +20.51 \text{ m.t.}$$

$$M_{bc} = -24 + 2\,(3.665) + (-3.838) = -20.51 \text{ m.t.}$$

$$M_{cb} = +24 + (3.665) + 2\,(-3.838) = +19.99 \text{ m.t.}$$

$$M_{cd} = 6\,(-3.838) - 13.5\,(-0.225) = -19.99 \text{ m.t.}$$

$$M_{dc} = 3\,(-3.838) - 13.5\,(-0.225) = -8.48 \text{ m.t.}$$

The B.M.D. is thus as shown in Fig. 5.16c.

Example 5.11 Construct the B.M.D. for the frame shown in Fig. 5.17a if the moment of inertia varies as indicated.

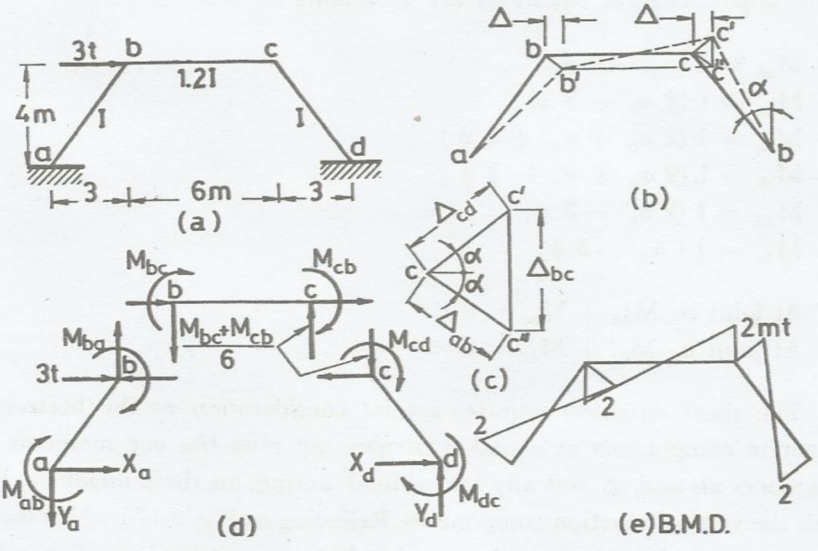


Fig. 5.17

Solution : In the previous examples on frames involving joint translation, all the members were normal to each other. This resulted in obvious relationships between various member rotations. In the given frame, and similar frames having inclined members, such relationships are not obvious and must be established. To do so, the members are imagined to be pin-connected. Since the length of each member does not change and the deflections are small, the displacement of one end of a member relative to its other end is normal to the axis of this member. This leads to the joint displacements indicated in Fig. 5.17b. It should be remembered that the dashed lines in this figure represent the chords of the members after deformation not the elastic curve of the frame. Triangle c c′c″, enlarged in Fig. 5.17c, shows the relationships between Δ_{ab}, Δ_{bc} and Δ_{cd}. Taking Δ as a reference value,

$$\Delta_{ab} = \Delta_{cd} = \Delta/\cos\alpha = 5\Delta/4 \text{ and } \Delta_{bc} = 2\Delta\tan\alpha = 3\Delta/2$$

$$\psi_{ab} = \psi_{cb} = 5\Delta/4\times 5 = \Delta/4 = \psi$$

$$\psi_{bc} = -3\Delta/2\times 6 = -\Delta/4 = -\psi$$

Note that ψ_{bc} is negative as member bc rotates anticlockwise.

Since no loads act on individual members, the fixed-end moments are all zero. The relative stiffnesses are as follows :

$$K_{ab} : K_{bc} : K_{cd} = I/5 : 1.2I/6 : I/5 = 1 : 1 : 1$$

The slope-deflection equations are as follows :

$$M_{ab} = 1\,(\alpha_b - 3\,\psi)$$
$$M_{ba} = 1\,(2\,\alpha_b - 3\,\psi)$$
$$M_{bc} = 1\,(2\,\alpha_b + \alpha_c + 3\,\psi)$$
$$M_{cb} = 1\,(2\,\alpha_c + \alpha_b + 3\,\psi)$$
$$M_{cd} = 1\,(2\,\alpha_c - 3\,\psi)$$
$$M_{dc} = 1\,(\alpha_c - 3\,\psi)$$

At joint b, $M_{ba} + M_{bc} = 0$

At joint c, $M_{ca} + M_{cd} = 0$

The shear equation requires special consideration as the horizontal reaction components at a and d involve not only the end moments on members ab and cd and any lateral load acting on these members, but also the vertical reaction components. Referring to Fig. 5.17d which shows the free-body diagram a nd by considering the equilibrium of member ab,

$$X_a = (M_{ab} + M_{ba})/4 - (M_{bc} + M_{cb})/8$$

Similarly, by considering the equilibrium of member cd,

$$X_d = (M_{cd} + M_{dc})/4 - (M_{bc} + M_{cb})/8$$

Substituting these values into the shear equations, $X_a + X_d + 3 = 0$,

$$(M_{ab} + M_{ba})/4 + (M_{cd} + M_{dc})/4 - (M_{bc} + M_{cb})/4 + 3 = 0$$

Substituting from the slope-deflection equations into the three equations of statics and simplifying,

$$4\,\alpha_b + \alpha_c = 0 \qquad \text{(i)}$$

$$\alpha_b + 4\,\alpha_c = 0 \qquad \text{(ii)}$$

$$3\,\psi = 2 \qquad \text{(iii)}$$

Solving equations, (i), (ii) and (iii) simultaneously, $\alpha_b = 0$, $\alpha_c = 0$ and $\psi = 2/3$. Substituting these values into the slope-deflection equations,

$$M_{ab} = M_{ba} = M_{cd} = M_{dc} = -2 \text{ m.t. and } M_{bc} = M_{cb} = 2 \text{ m.t.}$$

The final B.M.D. is thus as shown in Fig. 5.17e.

Example 5.12 Construct the B.M.D. for the frame shown in Fig. 5.18a. if the moment of inertia varies as indicated.

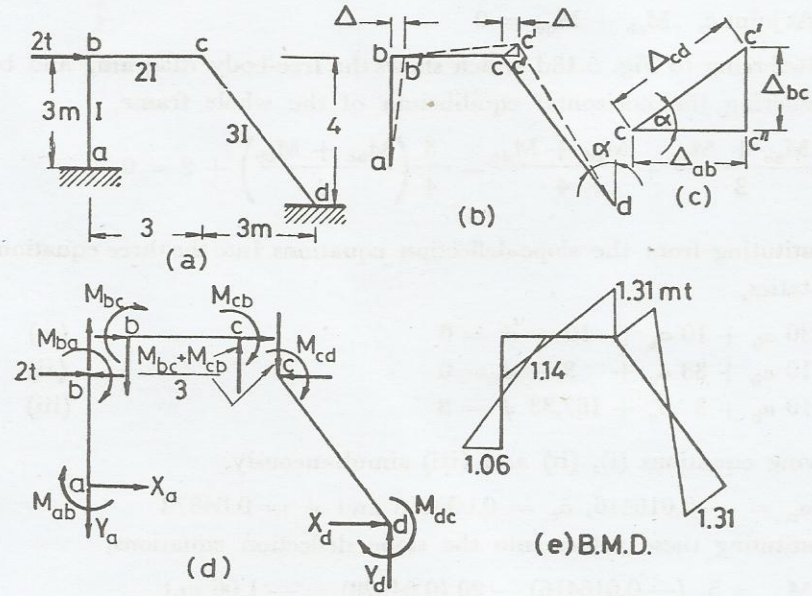


Fig. 5.18

Solution : The displacement diagram is shown in Fig. 5.18b. The relative displacements between the ends of each member are shown enlarged in Fig. 5.18c. Referring to this figure,

$$\Delta_{ab} = \Delta, \Delta_{bc} = \frac{\Delta}{\cos\ \alpha} = \frac{5}{4}\Delta \text{ and } \Delta_{bc} = \Delta \tan \alpha = \frac{3}{4}\Delta$$

$$\psi_{ab} : \psi_{bc} : \psi_{cd} = \frac{\Delta}{3} : \frac{-5\Delta}{4 \times 3} : \frac{3\Delta}{4 \times 5} = 1.33\psi : -\psi : \psi$$

Note that ψ_{bc} is negative as member bc rotates anticlockwise.

$$K_{ab} : K_{bc} : K_{cd} = \frac{I}{3} : \frac{2I}{3} : \frac{3I}{5} = 5 : 10 : 9$$

The slope deflection equations are as follows :

$$M_{ab} = 5\ (\alpha_b - 4\ \psi)$$
$$M_{ba} = 5\ (2\ \alpha_b - 4\ \psi)$$
$$M_{bc} = 10\ (2\ \alpha_b + \alpha_c + 3\ \psi)$$
$$M_{cb} = 10\ (\alpha_b + 2\ \alpha_c + 3\ \psi)$$

$M_{cd} = 9\,(2\,\alpha_c - 3\,\psi)$
$M_{dc} = 9\,(\alpha_c - 3\,\psi)$

At joint b, $M_{ba} + M_{bc} = 0$
At joint c, $M_{cb} + M_{cd} = 0$

Referring to Fig. 5.18d which shows the free-body diagram, and by considering the horizontal equilibrium of the whole frame,

$$\frac{M_{ab} + M_{bc}}{3} + \frac{M_{cd} + M_{dc}}{4} - \frac{3}{4}\left(\frac{M_{bc} + M_{cb}}{3}\right) + 2 = 0$$

Substituting from the slope-deflection equations into the three equations of statics,

$$30\,\alpha_b + 10\,\alpha_c + 10\,\psi = 0 \qquad \text{(i)}$$
$$10\,\alpha_b + 38\,\alpha_c + 3\,\psi = 0 \qquad \text{(ii)}$$
$$10\,\alpha_b + 3\,\alpha_c + 167.33\,\psi = 8 \qquad \text{(iii)}$$

Solving equations (i), (ii) and (iii) simultaneously,

$\alpha_b = -0.016416$, $\alpha_c = 0.000468$ and $\psi = 0.04878$

Substituting these values into the slope deflection equations,

$M_{ab} = 5\ (-0.016416) - 20\ (0.04878) = -1.06$ m.t.
$M_{ba} = 10\ (-0.016416) - 20\ (0.04878) = -1.14$ m.t.
$M_{bc} = 20\ (-0.016416) + 10\ (0.000468) + 30\ (0.04878) = +1.14$ m.t.
$M_{cb} = 10\ (-0.016416) + 20\ (0.000468) + 30\ (0.04878) = +1.31$ m.t.
$M_{cd} = 18\ (0.000468) - 27\ (0.04878) = -1.31$ m.t.
$M_{dc} = 9\ (0.000468) - 27\ (0.04878) = +1.31$ m.t.

The final B.M.D. is thus as shown in Fig. 5.18 e.

5.10 Applications to frames with multiple degrees of freedom in translation

The principles discussed in the preceding section in regard to frames with a single degree of freedom in translation are applicable to frames with any degree of freedom. In this case, however, there will be as many shear equations as there are independent joint translations. The difficulty associated with such problems arise mainly from setting the necessary shear equations and the simultaneous solution of the relatively large number of equations involved. Since solving of equations is of no concern in this book, only the slope-deflection equations and the necessary conditions of equilibrium will be given in the following examples.

Example 5.13 Write down, without solution, the slope-deflection equations and the conditions of equilibrium necessary for a complete analysis of the two-storey frame shown in Fig. 5.19. The relative stiffnesses of the various members are as indicated.

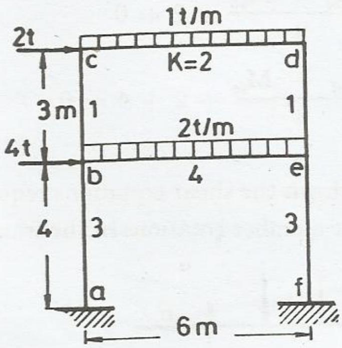


Fig. 5.19

Solution : From equations 5.1 and 5.2, the degree of freedom in rotation $s_r = 4$ and the degree of freedom in translation $s_t = 2$. Denoting the rotation of members ab and fe by ψ_1 and the rotation of members bc and ed by ψ_2 the slope-deflection equations will be as follows :

$$M_{ab} = 3\,(\alpha_b - 3\,\psi_1)$$
$$M_{ba} = 3\,(2\,\alpha_b - 3\,\psi_1)$$
$$M_{bc} = 1\,(2\,\alpha_b + \alpha_c - 3\,\psi_2)$$
$$M_{cb} = 1\,(2\,\alpha_c + \alpha_b - 3\,\psi_2)$$
$$M_{be} = -6 + 4\,(2\,\alpha_b + \alpha_e)$$
$$M_{eb} = +6 + 4\,(2\,\alpha_e + \alpha_b)$$
$$M_{cd} = -3 + 2\,(2\,\alpha_c + \alpha_d)$$
$$M_{dc} = +3 + 2\,(\,\alpha_d + \alpha_c)$$
$$M_{de} = 1\,(2\,\alpha_d + \alpha_e - 3\,\psi_2)$$
$$M_{ed} = 1\,(2\,\alpha_e + \alpha_d - 3\,\psi_2)$$
$$M_{ef} = 3\,(2\,\alpha_e - 3\,\psi_1)$$
$$M_{fe} = 3\,(\,\alpha_e - 3\,\psi_1)$$

The equations of equilibrium should be equal to the combined degree of freedom = 4 + 2 = 6. Four equations are provided by the joint equilibrium conditions at b,c,d and e, and two equations are provided by the horizontal equilibrium of each of the two storeys. Thus, the required conditions are :

$$M_{ba} + M_{bc} + M_{be} = 0 \qquad (i)$$
$$M_{cb} + M_{cd} = 0 \qquad (ii)$$
$$M_{dc} + M_{de} = 0 \qquad (iii)$$
$$M_{ed} + M_{eb} + M_{ef} = 0 \qquad (iv)$$
$$\frac{M_{bc} + M_{cb}}{3} + \frac{M_{de} + M_{ed}}{3} + 2 = 0 \qquad (v)$$
$$\frac{M_{ab} + M_{ba}}{4} + \frac{M_{ef} + M_{fe}}{4} + 2 + 4 = 0 \qquad (vi)$$

Example 5.14 Write down the shear equations required for the determination of the independent member rotations in the frame shown in Fig. 5.20a.

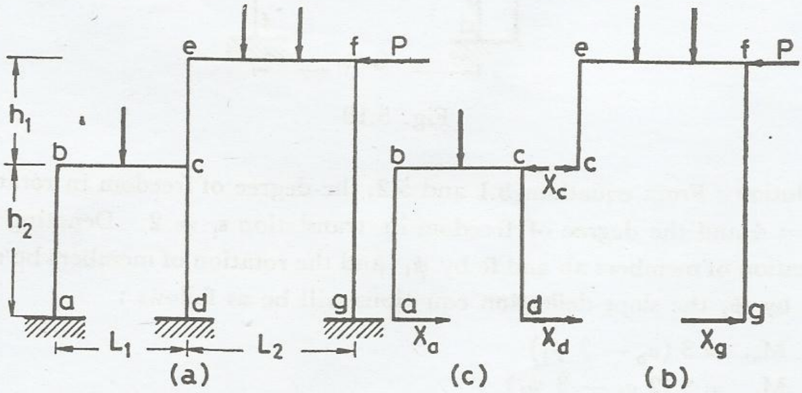


Fig. 5.20

Solution : From equation 5.2, the degree of freedom in translation $s_t = 2$. Joints a,d and g are fixed in position while joints b and c may move horizontally an amount Δ_1 and joints e and f an amount Δ_2.

$$\psi_{ab} = \psi_{cd} = \frac{\Delta_1}{h_1}, \quad \psi_{ec} = \frac{\Delta_2 - \Delta_1}{h_2} \text{ and } \psi_{fg} = \frac{\Delta_2}{(h_1 + h_2)}$$

Form 5.6 of the slope-deflection equations is to be used for all the vertical members and form 5.7 for the horizontal members. Further, it is necessary to have two shear equations in order to determine the two independent displacements Δ_1 and Δ_2.

Considering members ec and fg as free-bodies,

$$X_c = \frac{M_{ce} + M_{ec}}{h_1} \rightarrow \text{ and } X_g = \frac{M_{fg} + M_{gf}}{(h_1 + h_2)} \rightarrow$$

Referring to the free-body diagram of part cefg shown in Fig. 5.20b,

$$X_c + X_g - P = 0 \qquad \text{(i)}$$

Considering members ab and cd as free-bodies,

$$X_a = \frac{M_{ab} + M_{ba}}{h_2} \rightarrow \text{ and } X_d = \frac{M_{cd} + M_{dc}}{h_2} \rightarrow$$

Referring to the free-body diagram of part abcd shown in Fig. 5.20c,

$$X_a + X_d - X_c = 0 \qquad \text{(ii)}$$

The two shear equations (i) and (ii) in addition to the joint equilibrium equations are sufficient to determine the four unknown joint rotations; α_b, α_c, α_e and α_f and the two independent displacements Δ_1 and Δ_2.

5.11 Applications to statically indeterminate frames with support movements

The slope-deflection method is applicable to the analysis of statically indeterminate frames with known support movements. In this regard, form 5.4. of the slope-deflection equations is used. On so doing, the absolute values of member stiffnesses should be used and the known joint and member rotations should be expressed in radians. Further, it should be noticed that in the absence of the applied loads all the fixed-end moments are zero.

Consider, for example, the frame shown in Fig. 5.21, and let it be required to determine all the end moments resulting from a clockwise rotational slip at support a= θ and a vertical settlement at support d=Δ.

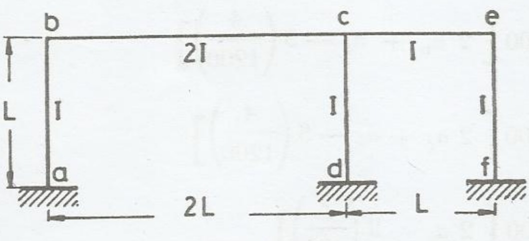


Fig. 5.21

Before writing up the slope-deflection equations, it is noticed that $\alpha_a = +\theta$ and that $\psi_{bc} = \Delta/2L$ and $\psi_{ce} = -\Delta/L$. Thus, one is left with three unknown joint rotations; α_b, α_c and α_e, and one unknown member rotation ψ ; $\psi_{ab} = \psi_{cd} = \psi_{ef} = \psi$. The values of these unknowns may be found from the three conditions of equilibrium at joints b, c and e and the shear equation.

Example 5.15 Analyse the frame shown in Fig. 5.16a in the absence of load if support a has a clockwise rotational slip = 0.002 rad. and support d settles by 4 cm. EI = 12000 m^2t.

Solution : The absolute (2 EI/L)-values of the various members are first evaluated and are as follows :

$$\text{member ab, } \frac{2\,EI}{L} = \frac{2 \times 12000}{6} = 4000 \text{ m.t.}$$

$$\text{member bc, } \frac{2\,EI}{L} = \frac{2 \times 12000}{12} = 2000 \text{ m.t.}$$

$$\text{member cd, } \frac{2\,EI}{L} = \frac{2 \times 12000}{4} = 6000 \text{ m.t.}$$

Member rotations :

$$\psi_{ab} = \frac{\Delta}{600}, \quad \psi_{bc} = \frac{4}{1200} \quad \text{and} \quad \psi_{cd} = \frac{\Delta}{400}$$

Here it is known that $\alpha_a = +0.002$ and $\alpha_d = 0$. Thus, the slope-deflection equations are as follows :

$$M_{ab} = 4000\left[2\,(0.002) + \alpha_b - 3\left(\frac{\Delta}{600}\right)\right]$$

$$M_{ba} = 4000\left[2\,\alpha_b + 0.002 - 3\left(\frac{\Delta}{600}\right)\right]$$

$$M_{bc} = 2000\left[2\,\alpha_b + \alpha_c - 3\left(\frac{4}{1200}\right)\right]$$

$$M_{cb} = 2000\left[2\,\alpha_c + \alpha_b - 3\left(\frac{4}{1200}\right)\right]$$

$$M_{cd} = 6000\left[2\,\alpha_c - 3\left(\frac{\Delta}{400}\right)\right]$$

$$M_{dc} = 6000\left[\alpha_c - 3\left(\frac{\Delta}{400}\right)\right]$$

Joint equilibrium equations :

At joint b, $M_{ba} + M_{bc} = 0$

At joint c, $M_{cb} + M_{cd} = 0$

Shear equation :

$$\frac{M_{ab} + M_{ba}}{6} + \frac{M_{cd} + M_{dc}}{4} = 0$$

Substituting from the slope-deflection equations into the three equations of statics,

$$600\ \alpha_b + 100\ \alpha_c - \quad \Delta = 0.6 \qquad \text{(i)}$$
$$100\ \alpha_b + 800\ \alpha_c - 2.25\ \Delta = 1.0 \qquad \text{(ii)}$$
$$-\alpha_b - 2.25\ \alpha_c + 0.01458\ \Delta = 0.002 \qquad \text{(iii)}$$

Solving equations (i), (ii) and (iii) simultaneously,

$$\alpha_b = 0.0017\ , \quad \alpha_c = 0.0031 \quad \text{and} \quad \Delta = 0.7316$$

$$M_{ab} = 4000\left[2\ (0.002) + 0.0017 - 3\left(\frac{0.7316}{600}\right)\right] = +\ 8.4\ \text{m.t.}$$

$$M_{ba} = 4000\left[2\ (0.0017) + 0.002 - 3\left(\frac{0.7316}{600}\right)\right] = +\ 7.0\ \text{m.t.}$$

$$M_{bc} = 2000\ [2\ (0.0017) + 0.0031 - 0.01] = -\ 7.0\ \text{m.t.}$$

$$M_{cb} = 2000\ [2\ (0.0031) + 0.0017 - 0.01] = -\ 4.2\ \text{m.t.}$$

$$M_{cd} = 6000\left[2\ (0.0031) - 3\left(\frac{0.7316}{400}\right)\right] = +\ 4.2\ \text{m.t.}$$

$$M_{dc} = 6000\left[0.0031 - 3\left(\frac{0.7316}{400}\right)\right] = -\ 14.4\ \text{m.t.}$$

5.12 Special simplifying technique

The application of the slope-deflection equations to the analysis of beams and frames has been general with no reference being made to any special procedures that may be taken to reduce the labour involved. Such procedures, however, are possible and exist when :

(1) hinged supports are present,

(2) there is symmetry in the structure.

The corresponding simplifications will be considered separately below :

(1) Modification for hinged ends

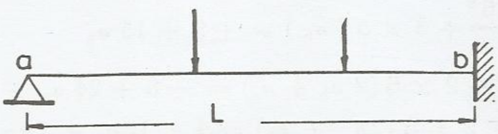


Fig. 5.22

Consider a member ab having a hinged support at a as shown in Fig. 5.22. The basic slope-deflection equations for this member are:

$$M_{ab} = M^F{}_{ab} + \frac{2\ EI}{L}(2\ \alpha_a + \alpha_b - 3\ \psi)$$

$$M_{ba} = M^F{}_{ba} + \frac{2\ EI}{L}(2\ \alpha_b + \alpha_a - 3\ \psi)$$

Since $M_{ab} = 0$, the first equation may be re-arranged to give :

$$\alpha_a = -M^F{}_{ab}\left(\frac{L}{4\ EI}\right) - \frac{\alpha_b}{2} + \frac{3}{2}\psi$$

Substituting this value of α_a into the second equation,

$$M_{ba} = \left(M^F_{ba} - \frac{M^F_{ab}}{2}\right) + \frac{3\ EI}{L}(\alpha_b - \psi) \qquad \dots\ 5.8$$

Equation 5.8 may be re-expressed as :

$$M_{ba} = \overline{M}^F_{ba} + \frac{3\ EI}{L}(\alpha_b - \psi) \qquad \dots\ 5.9$$

where $\overline{M}^F_{ba} = M^F_{ba} - M^F{}_{ab}/2$ ... 5.10

A study of equation 5.10 shows that $\overline{M}^F_{ba}$ represents the fixed-end moment at end b when end a is hinged. Also, M^F_{ab} adds to the numerical value of M^F_{ba}. For example if beam ab shown in Fig. 5.22 carries a uniformly distributed load, $\overline{M}^F_{ba} = \frac{wL^2}{12} - \left(-\frac{wL^2}{2 \times 12}\right) = \frac{wL^2}{8}$

Equation 5.9 expresses M_{ba} in terms of α_b and ψ without involving α_a. This results in one less unknown to be determined. To illustrate the simplification resulting from the application of equation 5.9, two problems previously solved using the basic slope-deflection equations will be considered.

Example 5.16 Re-analyse the beam in example 5.3(Fig. 5.6) using the modified slope-deflection equations.

Solution : The slope-deflection equations are :

$$M_{ba} = \frac{2 \times 6^2}{8} + 3 \times 5\ (\alpha_b) = +9 + 15\ \alpha_b$$

$$M_{bc} = -8 + 2 \times 6\ (2\ \alpha_b + \alpha_c) = -8 + 24\ \alpha_b + 12\ \alpha_c$$

$$M_{cb} = 4 + 2 \times 6\ (2\alpha_c + \alpha_b) = 4 + 12\ \alpha_b + 24\ \alpha_c$$

$$M_{cd} = -8 + 3 \times 5\ (\alpha_c) = -8 + 15\ \alpha_c$$

Two unknown rotations appear in the slope-deflection equations. These rotations may be determined from the equilibrium conditions at joints **b** and c; $M_{ba} + M_{bc} = 0$ and $M_{cb} + M_{cd} = 0$. Thus,

$$39\,\alpha_b + 12\,\alpha_c = -1 \qquad \text{(i)}$$

$$12\,\alpha_b + 39\,\alpha_c = 4 \qquad \text{(ii)}$$

Solving equations (i) and (ii) simultaneously, $\alpha_b = -0.063$ and $\alpha_c = 0.122$. Substituting these values into the slope-deflection equations,

$$M_{ba} = +9 + 15\,(-0.063) = +8.05 \text{ m.t.}$$

$$M_{bc} = -8 + 24\,(-0.063) + 12\,(0.122) = -8.05 \text{ m.t.}$$

$$M_{cb} = +4 + 12\,(-0.063) + 24\,(0.122) = 6.18 \text{ m.t.}$$

$$M_{cd} = -8 + 15\,(0.122) = -6.18 \text{ m.t.}$$

These moments are identical to those obtained by the lengthy solution presented in example 5.3.

Example 5.17 Re-analyse the frame in example 5.7 (Fig. 5.11) using the modified slope-deflection equations.

Solution : The slope-deflection equations are :

$$M_{ba} = +4.5 + 3 \times 4\,(\alpha_b)$$

$$M_{bc} = -16 + 2 \times 3\,(2\,\alpha_b)$$

$$M_{cb} = +16 + 2 \times 3\,(\alpha_b)$$

There is but one unknown rotation which may be determined from the equilibrium condition at joint b; $M_{ba} + M_{bc} = 0$. Thus,

$$-11.5 + 24\,\alpha_b = 0, \text{ from which } \alpha_b = \frac{11.5}{24} = 0.48$$

Substituting this value into the slope-deflection equations,

$$M_{ba} = +4.5 + 12\,(0.48) = +10.25 \text{ m.t.}$$

$$M_{bc} = -16 + 12\,(0.48) = -10.25 \text{ m.t.}$$

$$M_{cb} = +16 + 6\,(0.48) = +18.87 \text{ m.t.}$$

These values are identical to those obtained by the lengthy solution presented in example 5.7.

(2) Allowance for symmetry

When a structure is symmetrical, rotations of the various joints on one side of the axis of symmetry are related to those of the corresponding joints on the other side of the axis of symmetry. The type of loading then

deoides the nature of this relationship. Loads may be unsymmetrical, symmetrical, or antisymmetrical. Simplification results from the latter two cases of loading. This is illustrated with reference to Fig. 5.23.

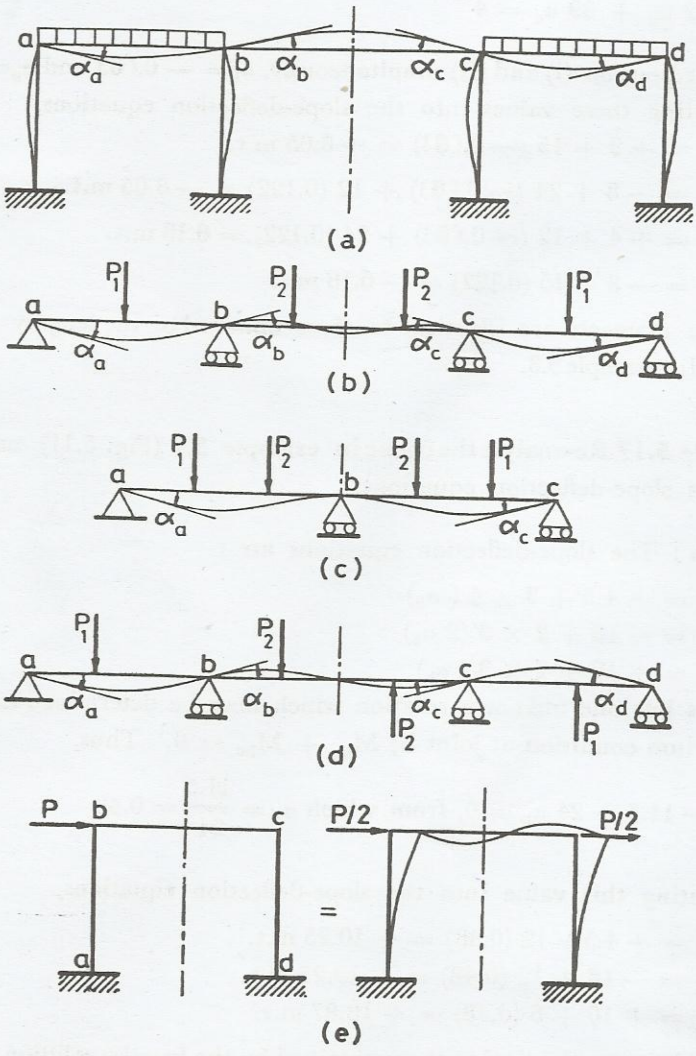


Fig. 5.23

Symmetry : A study of the elastic curve of the structures in Figs 5.23a and b, which have a centre span, shows that $\alpha_a = -\alpha_d$ and $\alpha_b = -\alpha_c$ and thus there are two unknown rotations only instead of four. In Fig. 5.23 c, which shows a structure with no centre span, $\alpha_a = -\alpha_c$

and $\alpha_{bl} = -\alpha_{br}$. Since, however, the elastic curve is continuous, the only way for α_{bl} to be numerically equal and of opposite sign to α_{br} is that both of them to be equal to zero. In other words, the given beam may be considered as one-span beam simply supported at one end and fixed at the other end.

Antisymmetry : It is seen in Fig. 5.23d that $\alpha_a = \alpha_d$ and $\alpha_b = \alpha_c$. Similarly, in Fig. 5.23e, $\alpha_b = \alpha_c$. Thus, in each of these two structures the number of unknown joint rotations is reduced by one half.

Unsymmetry : In general, loads are unsymmetrical, but any unsymmetrical load system may be divided into a symmetrical and an antisymmetrical load systems, to which the above simplifications apply. Consider, for example, the frame shown in Fig. 5.24a. The given loading is equivalent to the sum of the symmetrical loading shown in Fig. 5.24b and the antisymmetrical loading shown in Fig. 5.24c. The original frame involves the solution of six simultaneous equations in six unknown rotations; α_b,

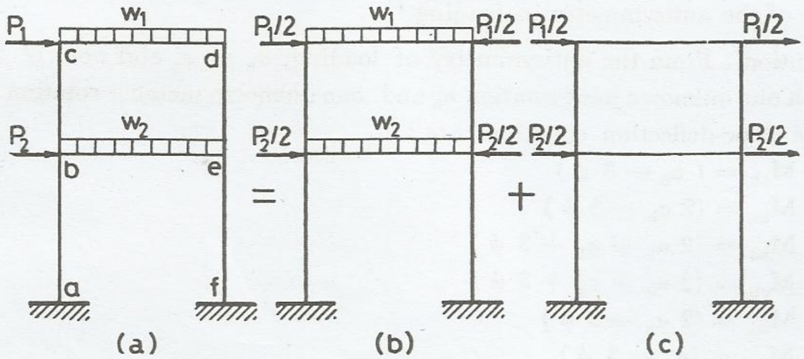


Fig. 5.24

α_c, α_d, α_e, ψ_{ab} $(= \psi_{fe})$ and ψ_{bc} $(= \psi_{ed})$. The frame under symmetrical loading involves the solution of two simultaneous equations in two unknown joint rotations; α_b $(= -\alpha_e)$ and α_c $(= -\alpha_d)$. The frame under the antisymmetrical loading involves the solution of four simultaneous equations in four unknown rotations; α_b $(= \alpha_e)$, α_c $(= \alpha_d)$, ψ_{ab} $(= \psi_{fe})$ and ψ_{bc} $(= \psi_{ed})$. Thus, the original problem which involves the solution of six equations is replaced by two component problems; the first involving the solution of two equations and the second four equations, and it is this procedure which causes a reduction in the computation time.

Example 5.18 Re-analyse the frame in example 5.6 (Fig. 5.10a) making use of the symmetry in the frame and loading.

Solution : From symmetry there is no joint translation and $\alpha_b = -\alpha_c$. The slope-deflection equations are :

$$M_{ab} = 0 + 4\,(\alpha_b) \qquad , M_{ab} = 4\,\alpha_b$$
$$M_{ba} = 0 + 4\,(\alpha_b) \qquad , M_{ba} = 8\,\alpha_b$$
$$M_{bc} = -13 + 5\,(2\,\alpha_b - \alpha_b) \qquad , M_{bc} = -13 + 5\,\alpha_b$$

These moments are expressed in terms of a single rotation α_b. This is found from the equilibrium condition at joint b, $M_{ba}+M_{bc}=0$. Thus,

$8\,\alpha_b - 13 + 5\,\alpha_b = 0$, from which $\alpha_b = 1$.

By back-substitution, into the slope-deflection equations, $M_{ab} = +4$, $M_{ba} = +8$ and $M_{bc} = -8$ m.t.

These moments are identical to those previously obtained in example5.6

Example 5.19 Re-analyse the frame in example 5.11 (Fig. 5.17a) making use of the antisymmetry in loading.

Solution : From the antisymmetry of loading, $\alpha_b = \alpha_c$ and one is left with one unknown joint rotation α_b and one unknown member rotation ψ.

The slope-deflection equations are :

$$M_{ab} = (\alpha_b - 3\,\psi)$$
$$M_{ba} = (2\,\alpha_b - 3\,\psi)$$
$$M_{bc} = (2\,\alpha_b + \alpha_b + 3\,\psi)$$
$$M_{cb} = (2\,\alpha_b + \alpha_b + 3\,\psi)$$
$$M_{cd} = (2\,\alpha_b - 3\,\psi)$$
$$M_{dc} = (\alpha_b - 3\,\psi)$$

The two unknown rotations involved are determined from the equilibrium condition at joint b and the shear equation.

$$M_{ba} + M_{bc} = 0$$

$$\frac{M_{ab} + M_{ba}}{4} + \frac{M_{cd} + M_{dc}}{4} - \frac{M_{bc} + M_{cb}}{4} + 3 = 0$$

Substituting from the slope-deflection equations into the two equations of statics and solving simultaneously, $\alpha_b = 0$ and $\psi = 2/3$.

Substituting these values into the slope-deflection equations,

$$M_{ab} = M_{ba} = M_{cd} = M_{dc} = -2 \text{ m.t. and } M_{bc} = M_{cb} = 2 \text{ m.t}$$

These moments are identical to those obtained by the lengthy solution presented in example 5.11.

EXAMPLES TO BE WORKED OUT

(1) - (11) Using the method of slope-deflection draw the B.M.D. for each of the beams shown in Figs. 3.32-3.42. EI = constant.

(12)-(17) Re-solve problems 1, 2, 3, 5, 7, 9 if the moment of inertia varies from span to span as follows :

Problem 1 (Fig. 3.32), $I_{ab} : I_{bc} : I_{cd} = 2 : 1.2 : 1$

Problems 2 and 3 (Figs. 3.33 and 3.34), $I_{ab} : I_{bc} : I_{cd} = 1.5 : 2 : 1$

Problems 5 and 7 (Figs. 3.36 and 3.38), $I_{ab} : I_{bc} = 1.5 : 1$

Problem 9 (Fig. 3.40), $I_{ab} : I_{bc} : I_{cd} = 1 : 2 : 1$

(18)-(20) Determine the moments at the supports of the beams in problems 1, 4 and 6 in the absence of the applied loads due to the following support movements :

Problem 1 (Fig. 3.32), $\delta_b = 2$ cm. and $\delta_c = 0.5$ cm. EI = 4000 m^2t.

Problem 4 (Fig. 3.35), i — $\delta_b = 2.5$ cm.
ii — $\alpha_a = 0.005$ cm. EI = 4000 m^2t.

Problem 6 (Fig. 3.37), i — $\alpha_c = 0.0045$ rad. anticlockwise.
ii — $\delta_b = 3.6$ cm. EI = 5000 m^2t.

(21)-(24) Using the slope-deflection method, draw the B.M.D. for each of the frames shown in Figs. 3.45-3.57 if the moment of inertia varies as indicated.

(25)-(42) Using the slope-deflection method, draw the B.M.D. for each of the frames shown in Figs. 4.23, 4.24, 4.25, 4.27, 4.28, 4.30, 4.32 and 4.34 if the moment of inertia varies as indicated.

(43)-(44) Using the slope-deflection method, draw the B.M.D. for each of the frames in Figs. 5.25 and 5.26. Check your results by any convenient method.

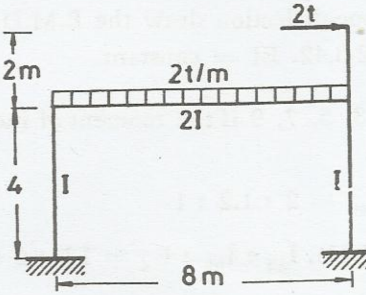


Fig. 5.25

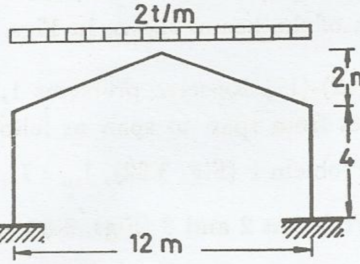


Fig. 5.26

(45)-(52) State the degrees of freedom in rotation and translation of each of the frames shown in Figs. 5.27-5.34, then write down the equations of statics (in terms of member end moments) which are necessary for the determination of the independent joint and member rotations.

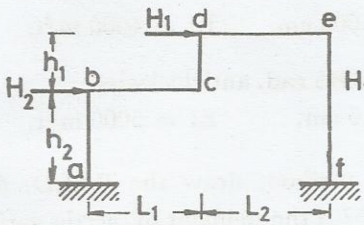


Fig. 5.27

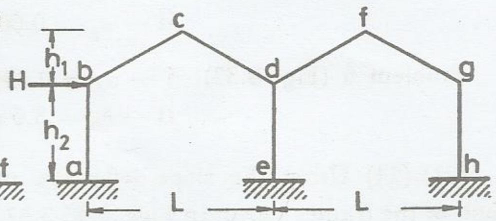


Fig. 5.28

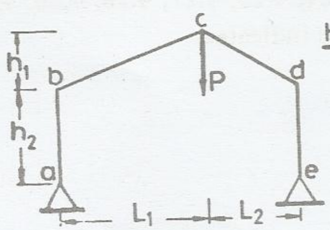


Fig. 5.29

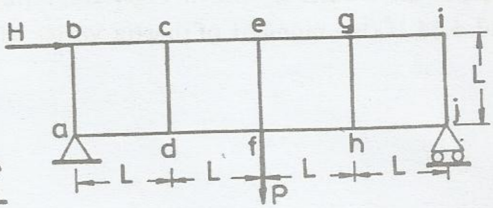


Fig. 5.30

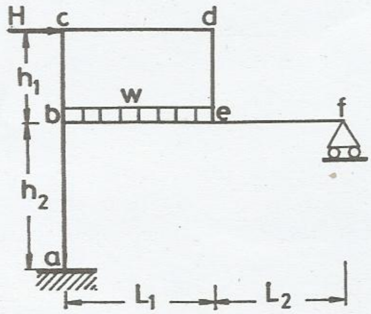


Fig. 5.31

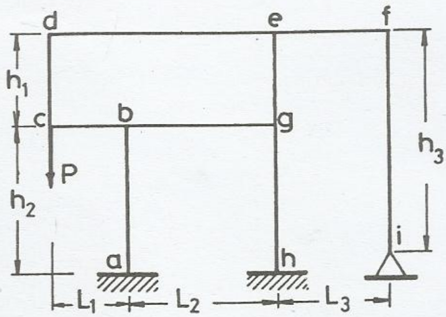


Fig. 5.32

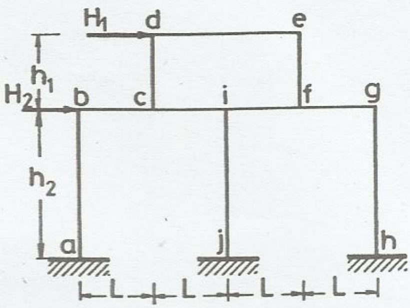


Fig. 5.33

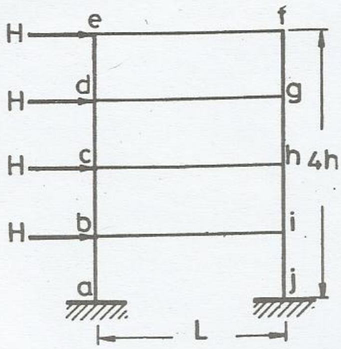


Fig. 5.34