COMBINATIONS WITH REPETITIONS

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Given n distinct objects and asked to select r objects, the number of such selections (order is not important) or combinations is $nCr = \binom{n}{r}$

For example: $S = \{a, b, c\}$, $n = |S| = 3$. Select $r = 2$.

There are ${}_3C_2 = 3$ combinations. They are: { a, b}, {a, c}, {b, c}.

Now assume that there are unlimited repetitions of each of the n distinct types of objects.

How many different selections of r objects (order is not important) are there?

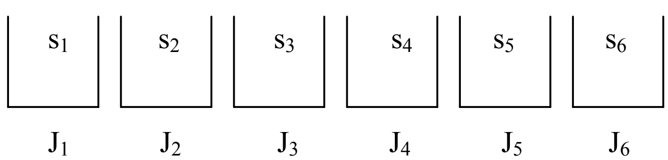
For example S= {a, b, c} where there are unlimited number of a's, unlimited number of b's, unlimited number of c's. In this case, to be more clear, we shall say that we have n = 3 categories of objects.

Now the different combinations or selections (order is not important) with repetitions of r = 2 objects from these n = 3 categories are

{a,b}, {a,c}, {b,c}, {a, a}, {b, b}, {c, c}

The formula is ${}_{n+r-1}C_r = {}_{3+2-1}C_2 = {}_4C_2 = 6$. Let us derive this formula using an example.

Assume there are n = 6 jars denoted by J_i. Each jar



contains unlimited number of a particular type of sweet s_i. So there are n = 6 categories of sweets. We are asked to select r = 10 sweets from these n = 6 jars of sweets.

How many such combinations with repetitions are there?

To begin with we observe that

- The order of the jars is not important.
- The order of the sweets is not important.

Any selection or combination of r = 10 sweets will do.

$$1s_1 + 1s_2 + 2s_3 + 3s_4 + 2s_5 + 1s_6$$

is such a combination or selection. How many such combinations are there?

Since we cannot solve the problem directly, we are going to translate this problem into another problem.

Distribution of r = 10 identical objects into n = 6 jars

Let us rewrite our selection of r = 10 sweets from n = 6 categories as

$$s_1 + s_2 + s_3\, s_3 + s_4\, s_4\, s_4 + s_5\, s_5 + s_6$$

Since any selection of r = 10 sweets will do, we do not care about the type of sweets selected. So we can say that the selected sweets are *identical*. Hence we may drop the subscripts. So the selection is now

$$s + s + s\, s + s\, s\, s + s\, s + s$$

We may now view the problem as *distributing* r = 10 *identical* objects into n = 6 jars. Again the order of the jars is not important.

Any such *distribution* of r = 10 *identical* objects is a combination or selection in the original problem. And any combination in the original problem is a *distribution* of *identical* objects of this problem. In other words the two problems are equivalent.

Let us see a few examples to make this equivalence of the two problems clear.

Selection: $0s_1 + 1s_2 + 2s_3 + 3s_4 + 2s_5 + 2s_6$

Distribution of identical: + s + s s + s s s + s s + s s

Selection: $0s_1 + 0s_2 + 0s_3 + 0s_4 + 0s_5 + 10s_6$

Meaning all r = 10 sweets were selected from jar 6.

Distribution of identical: + + + + + s s s s s s s s s s

Meaning all r = 10 sweets were distributed into jar 6.

Selection: $1s_1 + 1s_2 + 0s_3 + 0s_4 + 4s_5 + 4s_6$

Distribution of identical: s + s + + + s s s s + s s s s

How can we count the number of *distributions* of r = 10 *identical* objects into n = 6 jars? Since we cannot solve the problem directly, we are going to translate this problem into another problem.

Permutation with Repetitions problem

If we look at the *distribution* of $r = 10$ *identical* objects into $n = 6$ jars

$$s + s + s\,s + s\,s\,s + s\,s + s$$

we note that there are $r = 10$ symbols **'s'** and $(n - 1) = (6 - 1) = 5$ **'plus'** signs.

In addition to the sweets being *identical*, we also note that, since the order of the jars is not important, any permutation of these $(r + n - 1)$ symbols is also a *distribution* of $r =10$ *identical* objects into $n = 6$ jars or categories.

So now the *distribution* of $r = 10$ *identical* objects into $n = 6$ jars (where the order of the jars is not important) becomes a **Permutation with Repetitions** problem.

There are $(r + n - 1)$ symbols with $(n - 1)$ of one type (the **'plus'** signs) and r of another type (the **'s'** for identical sweets). For this we have a formula:

$$\frac{(r + n - 1)!}{(n - 1)!\, r!}$$

The formula may be written using the combination notation as

$${}_{n+r-1}C_r \text{ or } {}_{n+r-1}C_{n-1}$$

The number of ways to select r objects from n categories.

Number of non-negative integer solutions to an equation

There is another application of this formula. How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$$

Here there are $n = 6$ jars and $r = 10$ units. There are ${}_{n+r-1}C_r$ solutions.

We may vary this problem by imposing some restrictions below. For example:

How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$$

$$\text{with } x_1 \geq 2 \text{ and } x_2 \geq 1$$

With a change of variable trick we can convert this problem to a non-negative integer solutions problem.

$$y_1 = x_1 - 2,\ y_2 = x_2 - 1,\ y_3 = x_3,\ y_4 = x_4,\ y_5 = x_5 \text{ and } y_6 = x_6$$

So $x_1 = y_1 + 2$ and $x_2 = y_2 + 1$

Now: $(y_1 + 2) + (y_2 + 1) + y_3 + y_4 + y_5 + y_6 = 10$

For $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 10 - 2 - 1 = 7$

find the number of non-negative integer solutions with $n = 6$ and $r = 7$.